Partial Differential Equations

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Notations

Symbols

Δ $\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$

∇ $\left(\frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_n} \right).$

Ω denotes an open subset of $\mathbb{R}^n$, not necessarily bounded

$\partial\Omega$ denotes the boundary of Ω

$\mathbb{R}^n$ denotes the n-dimensional Euclidean space

D^α $\frac{\partial^{\alpha_1}}{\partial {x_1}^{\alpha_1}} \dots \frac{\partial^{\alpha_n}}{\partial {x_n}^{\alpha_n}}$ and $\alpha = (\alpha_1, \dots, \alpha_n)$

Function Spaces

$C(\Omega)$ is the class of all continuous functions on Ω

$C^k(\Omega)$ is the class of all k-times ($k \geq 1$) continuously differentiable functions on Ω

$C^\infty(\Omega)$ is the class of all infinitely differentiable functions on Ω

$C_c^\infty(\Omega)$ is the class of all infinitely differentiable functions on Ω with compact support

General Conventions

$B_r(x)$ denotes the open disk with centre at x and radius r

$S_r(x)$ denotes the circle or sphere with centre at x and radius r

w_n denotes the surface area of a n-dimensional sphere of radius 1.

Chapter 1

Introduction

A partial differential equation (PDE) is an equation involving an unknown function u of two or more variables and some or all of its partial derivatives.

Let Ω be an open subset of $\mathbb{R}^n$ and let $u : \Omega \to \mathbb{R}$ be a given function. We denote the partial derivative of u along $e_i = (0, 0, \ldots, 1, 0, \ldots, 0)$, the i-th coordinate vector of $\mathbb{R}^n$, as

$$u_{x_i} = \frac{\partial u}{\partial x_i}(x) = \lim_{h \to 0} \frac{u(x + he_i) - u(x)}{h},$$

provided the limit exists. Similarly, one can consider higher order derivatives, as well. We now introduce Schwartz's multi-index notation for derivative, which will be used to denote a PDE in a concise form. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an n-tuple of non-negative integers and let $|\alpha| = \alpha_1 + \ldots + \alpha_n$. The partial differential operator of *order* α is denoted as,

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial {x_1}^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial {x_n}^{\alpha_n}} = \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} \ldots \partial {x_n}^{\alpha_n}}.$$

If α and β are two multi-indices, then $\alpha \leq \beta$ means $\alpha_i \leq \beta_i$ for all $1 \leq i \leq n$. Also, $\alpha \pm \beta = (\alpha_1 \pm \beta_1, \ldots, \alpha_n \pm \beta_n)$, $\alpha! = \alpha_1! \ldots \alpha_n!$ and $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. For $k \geq 1$, we define $D^k u(x) := \{D^\alpha u(x) \mid |\alpha| = k\}$. Thus, for $k = 1$, we regard Du as being arranged in a vector, i.e.,

$$Du = (u_{x_1}, \ldots u_{x_n}).$$

We call this the *gradient* vector denoted as,

$$\nabla := \left(\frac{\partial}{\partial x_1}, \ldots \frac{\partial}{\partial x_n}\right).$$

Similarly, for $k = 2$, we regard D^2 as being arranged in a matrix form (called the *Hessian* matrix),

$$D^2 = \begin{pmatrix} \frac{\partial^2}{\partial x_1^2} & \cdots & \frac{\partial^2}{\partial x_1 \partial x_n} \\ \frac{\partial^2}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2}{\partial x_2 \partial x_n} \\ & \ddots & \\ \frac{\partial^2}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2}{\partial x_n^2} \end{pmatrix}_{n \times n}.$$

The trace of the Hessian matrix is called the Laplace operator, denoted as $\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. Note that under the order introduced on multi-indices α, $D^k u(x)$ can be regarded as a vector in R^{n^k}.

1.1 History of PDE

The study of partial differential equations started as a tool to analyse the models of physical science. The PDE's usually arise from the physical laws such as balancing forces (Newton's law), momentum, conservation laws etc. The first PDE was introduced in 1752 by d'Alembert as a model to study vibrating strings. He introduced the one dimensional *wave* equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2}.$$

This was then generalised to two and three dimensions by Euler (1759) and D. Bernoulli (1762), i.e.,

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \Delta u(x,t),$$

where $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$.

In physics, a *field* is a physical quantity associated to each point of space-time. A field can be classified as a scalar field or a vector field according to whether the value of the field at each point is a scalar or a vector, respectively. Some examples of field are Newton's gravitational field, Coulomb's electrostatic field and Maxwell's electromagnetic field.

Given a vector field V, it may be possible to associate a scalar field u, called *potential*, such that $\nabla u = V$. Moreover, the gradient of any function u, ∇u is a vector field. In gravitation theory, the gravity potential is the potential energy per unit mass. Thus, if E is the potential energy of an object with mass m, then $u = E/m$ and the potential associated with a mass distribution is the superposition of potentials of point masses.

The Newtonian gravitation potential can be computed to be

$$u(x) = \frac{1}{4\pi} \int_{\Omega} \frac{\rho(y)}{|x-y|}\, dy$$

where $\rho(y)$ is the density of the mass distribution, occupying $\Omega \subset \mathbb{R}^3$, at y. In 1782, Laplace discovered that the Newton's gravitational potential satisfies the equation:

$$\Delta u = 0 \quad \text{ on } \mathbb{R}^3 \setminus \overline{\Omega}.$$

Thus, the operator $\Delta = \nabla \cdot \nabla$ is called the *Laplacian* and any function whose Laplacian is zero (as above) is said to be a *harmonic* function.

Later, in 1813, Poisson discovered that on Ω the Newtonian potential satisfies the equation:

$$-\Delta u = \rho \quad \text{on } \Omega.$$

Such equations are called *Poisson* equation. The identity obtained by Laplace was, in fact, a consequence of the conservation laws and can be generalised to any scalar potential. Green (1828) and Gauss (1839) observed that the Laplacian and Poisson equations can be applied to any scalar potential including electric and magnetic potentials. Suppose there is a scalar potential u such that $V = \nabla u$ for a vector field V and V is such that $\int_{\partial\gamma} V \cdot \nu \, d\sigma = 0$ for all closed surfaces $\partial\gamma \subset \Gamma$. Then, by Gauss divergence theorem[1] (cf. Appendix C), we have

$$\int_{\gamma} \nabla \cdot V \, dx = 0 \quad \forall \gamma \subset \Gamma.$$

Thus, $\nabla \cdot V = \text{div} V = 0$ on Γ and hence $\Delta u = \nabla \cdot (\nabla u) = \nabla \cdot V = 0$ on Γ. Thus, the scalar potential is a harmonic function. The study of potentials in physics is called *Potential Theory* and, in mathematics, it is called Harmonic Analysis. Note that, for any potential u, its vector field $V = \nabla u$ is *irrotational*, i.e., $\text{curl}(V) = \nabla \times V = 0$.

Later, in 1822 J. Fourier on his work on heat flow in *Théorie analytique de la chaleur* introduced the heat equation

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t),$$

[1] a mathematical formulation of conservation laws

where $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$. The heat flow model was based on Newton's law of cooling.

Thus, by the beginning of 19th century, the three most important PDE's were identified.

1.2 Continuity Equation

Let us consider an ideal compressible fluid (viz. gas) occupying a bounded region $\Omega \subset \mathbb{R}^n$ (in practice, we take $n = 3$, but the derivation is true for all dimensions). For mathematical precision, we assume Ω to be a bounded open subset of $\mathbb{R}^n$. Let $\rho(x,t)$ denote the density of the fluid for $x \in \Omega$ at time $t \in I \subset \mathbb{R}$, for some open interval I. Mathematically, we presume that $\rho \in C^1(\Omega \times I)$. We cut a region $\Omega_t \subset \Omega$ and follow Ω_t, the position at time t, as t varies in I. For mathematical precision, we will assume that Ω_t have C^1 boundaries (cf. Appendix C). Now, the law of conservation of mass states that during motion the mass is conserved and mass is the product of density and volume. Thus, the mass of the region as a function of t is constant and hence its derivative should vanish. Therefore,

$$\frac{d}{dt} \int_{\Omega_t} \rho(x,t)\, dx = 0.$$

We regard the points of Ω_t, say $x \in \Omega_t$, following the trajectory $x(t)$ with velocity $\mathbf{v}(x,t)$. We also assume that the deformation of Ω_t is smooth, i.e., $\mathbf{v}(x,t)$ is continuous in a neighbourhood of $\Omega \times I$. Consider

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega_t} \rho(x,t)\, dx &= \lim_{h \to 0} \frac{1}{h} \left(\int_{\Omega_{t+h}} \rho(x,t+h)\, dx - \int_{\Omega_t} \rho(x,t)\, dx \right) \\
&= \lim_{h \to 0} \int_{\Omega_t} \frac{\rho(x,t+h) - \rho(x,t)}{h}\, dx \\
&\quad + \lim_{h \to 0} \frac{1}{h} \left(\int_{\Omega_{t+h}} \rho(x,t+h)\, dx - \int_{\Omega_t} \rho(x,t+h)\, dx \right)
\end{aligned}$$

The first integral becomes

$$\lim_{h \to 0} \int_{\Omega_t} \frac{\rho(x,t+h) - \rho(x,t)}{h}\, dx = \int_{\Omega_t} \frac{\partial \rho}{\partial t}(x,t)\, dx.$$

The second integral reduces as,

$$\begin{aligned}\int_{\Omega_{t+h}} \rho(x,t+h)\,dx - \int_{\Omega_t} \rho(x,t+h)\,dx &= \int_{\Omega} \rho(x,t+h)\left(\chi_{\Omega_{t+h}} - \chi_{\Omega_t}\right) \\ &= \int_{\Omega_{t+h}\setminus\Omega_t} \rho(x,t+h)\,dx \\ &\quad - \int_{\Omega_t\setminus\Omega_{t+h}} \rho(x,t+h)\,dx.\end{aligned}$$

We now evaluate the above integral in the sense of Riemann. We fix t. Our aim is to partition the set $(\Omega_{t+h}\setminus\Omega_t)\cup(\Omega_t\setminus\Omega_{t+h})$ with cylinders and evaluate the integral by letting the cylinders as small as possible. To do so, we choose $0 < s \ll 1$ and a polygon that covers $\partial\Omega_t$ from outside such that the area of each of the face of the polygon is less than s and the faces are tangent to some point $x_i \in \partial\Omega_t$. Let the polygon have m faces. Then, we have $x_1, x_2, \ldots x_m$ at which the faces $F_1, F_2, \ldots, F_m$ are a tangent to $\partial\Omega_t$. Since Ω_{t+h} is the position of Ω_t after time h, any point $x(t)$ moves to $x(t+h) = \mathbf{v}(x,t)h$. Hence, the cylinders with base F_i and height $\mathbf{v}(x_i,t)h$ is expected to cover our annular region depending on whether we move inward or outward. Thus, $\mathbf{v}(x_i,t)\cdot\nu(x_i)$ is positive or negative depending on whether Ω_{t+h} moves outward or inward, where $\nu(x_i)$ is the unit outward normal at $x_i \in \partial\Omega_t$.

$$\int_{\Omega_{t+h}\setminus\Omega_t} \rho(x,t+h)\,dx - \int_{\Omega_t\setminus\Omega_{t+h}} \rho(x,t+h)\,dx = \lim_{s\to 0}\sum_{i=1}^{m} \rho(x_i,t)\mathbf{v}(x_i,t)\cdot\nu(x_i)hs.$$

Thus,

$$\lim_{h\to 0}\frac{1}{h}\left(\int_{\Omega_{t+h}} \rho(x,t+h)\,dx - \int_{\Omega_t} \rho(x,t+h)\,dx\right) = \int_{\partial\Omega_t} \rho(x,t)\mathbf{v}(x,t)\cdot\nu(x)\,d\sigma.$$

By Green's theorem (cf. Appendix C), we have

$$\frac{d}{dt}\int_{\Omega_t} \rho(x,t)\,dx = \int_{\Omega_t}\left(\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{v})\right)dx.$$

Now, using conservation of mass, we get

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho v) = 0 \text{ in } \Omega\times\mathbb{R}. \tag{1.2.1}$$

Equation (1.2.1) is called the equation of continuity. In fact, any quantity that is conserved as it moves in an open set Ω satisfies the equation of continuity (1.2.1).

1.3 Definition and Well-Posedness of PDE

Definition 1.3.1. *Let Ω be an open subset of $\mathbb{R}^n$. A k-th* order *PDE F is a given map $F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \dots \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ having the form*

$$F\left(D^k u(x), D^{k-1}u(x), \dots Du(x), u(x), x\right) = 0, \tag{1.3.1}$$

for each $x \in \Omega$ and $u : \Omega \to \mathbb{R}$ is the unknown. We say F is linear *if* (1.3.1) *has the form*

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u(x) = f(x)$$

for given functions f and a_α ($|\alpha| \leq k$). If $f \equiv 0$, we say F is homogeneous. F is said to be semilinear*, if it is linear in the highest order, i.e., F has the form*

$$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u(x) + a_0(D^{k-1}u, \dots, Du, u, x) = 0.$$

We say F is quasilinear *if it has the form*

$$\sum_{|\alpha| = k} a_\alpha(D^{k-1}u(x), \dots, Du(x), u(x), x) D^\alpha u + a_0(D^{k-1}u, \dots, Du, u, x) = 0.$$

Finally, we say F is fully nonlinear *if it depends nonlinearly on the highest order derivatives.*

A problem involving a PDE could be a boundary-value problem (we look for a solution with prescribed boundary value) or a initial value problem (a solution whose value at initial time is known). Practically, it is usually desirable to solve a *well-posed* problem, in the sense of Hadamard . By well-posedness we mean that the PDE along with the boundary condition (or initial condition)

(a) has a solution (existence)

(b) the solution is unique (uniqueness)

(c) the solution depends continuously on the data given (stability).

Any PDE not meeting the above criteria is said to be *ill-posed*. Hadamard gave an example of an ill-posed problem.

1.4 Classification of PDE

1.4.1 Classification of Second order Linear PDE

Consider the second order linear PDE in two variables $(x, y) \in \mathbb{R}^2$,

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} = D(x,y,u,u_x,u_y) \tag{1.4.1}$$

where u, u_x, u_y appear linearly in the function D. Also, one of the coefficients A, B or C is identically non-zero (to make the PDE second order). The form above is a reformulation of a linear PDE, by accumulating the higher (second) order derivatives on one side. We observe that the choice of the independent variables (x, y) could affect the structure of the PDE. Let $w(x, y)$, $z(x, y)$ be a new pair of independent variable such that w, z are both continuous and twice differentiable w.r.t (x, y). We also assume that the Jacobian J,

$$J = \begin{pmatrix} w_x & w_y \\ z_x & z_y \end{pmatrix} \neq 0,$$

because a nonvanishing Jacobian ensures the existence of a one-to-one transformation between (x, y) and (w, z). Thus, we have

$$\begin{aligned}
u_x &= u_w w_x + u_z z_x, \\
u_y &= u_w w_y + u_z z_y, \\
u_{xx} &= u_{ww} w_x^2 + 2u_{wz} w_x z_x + u_{zz} z_x^2 + u_w w_{xx} + u_z z_{xx} \\
u_{yy} &= u_{ww} w_y^2 + 2u_{wz} w_y z_y + u_{zz} z_y^2 + u_w w_{yy} + u_z z_{yy} \\
u_{xy} &= u_{ww} w_x w_y + u_{wz}(w_x z_y + w_y z_x) + u_{zz} z_x z_y + u_w w_{xy} + u_z z_{xy}
\end{aligned}$$

Substituting above equations in (1.4.1), we get

$$a(w,z)u_{ww} + 2b(w,z)u_{wz} + c(w,z)u_{zz} = d(w,z,u,u_w,u_z).$$

where D transforms in to d and

$$\begin{aligned}
a(w,z) &= Aw_x^2 + 2Bw_x w_y + Cw_y^2 \\
b(w,z) &= Aw_x z_x + B(w_x z_y + w_y z_x) + Cw_y z_y \\
c(w,z) &= Az_x^2 + 2Bz_x z_y + Cz_y^2.
\end{aligned}$$

Note that the cofficients in the new coordinate system satisfy

$$b^2 - ac = (B^2 - AC)J^2.$$

Since $J \neq 0$, the change of variable leaves unchanged the sign of the *discriminant*, $b^2 - ac$ and $B^2 - AC$, of the PDE in each coordinate. Thus, the sign of the discriminant is invariant under coordinate transformation for which $J \neq 0$. Thus, we classify PDE based on the sign of its discriminant $d = B^2 - AC$. We say a PDE is of *hyperbolic* type if $d > 0$, *parabolic* type if $d = 0$ and *elliptic* type if $d < 0$. The motivation for these names are no indication of the geometry of the solution of the PDE, but just a correspondence with the corresponding second degree algebraic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Let $d = B^2 - AC$ be the discriminant of the algebraic equation and the the curve represented by the equation is a hyperbola if $d > 0$, parabola if $d = 0$ and ellipse if $d < 0$.

Observe that the classification of PDE is dependent on its coeficients, which may vary from region to region. Thus, for constant coefficients, the PDE remains unchanged throughout the region. However, for variable coefficients, the PDE may change its classification from region to region. An example is the *Tricomi* equation

$$u_{xx} + xu_{yy} = 0.$$

The discriminant of the Tricomi equation is $d = -x$. Thus, tricomi equation is hyperbolic when $x < 0$, elliptic when $x > 0$ and degenerately parabolic when $x = 0$, i.e., y-axis.

1.4.2 Standard or Canonical Forms of Second Order PDE

The advantage of above classification helps us in reducing a given PDE into simple forms. Given a PDE, one can compute the sign of the discriminant and depending on its clasification we can choose a coordinate transformation (w, z) such that $a = c = 0$ for hyperbolic, $a = b = 0$ for parabolic and $a = c, b = 0$ for elliptic type.

If the given second order PDE (1.4.1) is such that $A = C = 0$, then (1.4.1) is of hyperbolic type and a division by $2B$ (since $B \neq 0$) gives

$$u_{xy} = \tilde{D}(x, y, u, u_x, u_y)$$

where $\tilde{D} = D/2B$. The above form is the *first standard form* of second order hyperbolic equation. If a hyperbolic PDE is given in its first standard no coordinate transformation is required to simplify the PDE. If we introduce the linear change of variable $X = x + y$ and $Y = x - y$ in the first standard form, we get the *second standard form* of hyperbolic PDE

$$u_{XX} - u_{YY} = \hat{D}(X, Y, u, u_X, u_Y).$$

If the given second order PDE (1.4.1) is such that $A = B = 0$, then (1.4.1) is of parabolic type and a division by C (since $C \neq 0$) gives

$$u_{yy} = \tilde{D}(x, y, u, u_x, u_y)$$

where $\tilde{D} = D/C$. The above form is the *standard form* of second order parabolic equation.

If the given second order PDE (1.4.1) is such that $A = C$ and $B = 0$, then (1.4.1) is of elliptic type and a division by A (since $A \neq 0$) gives

$$u_{xx} + u_{yy} = \tilde{D}(x, y, u, u_x, u_y)$$

where $\tilde{D} = D/A$. The above form is the *standard form* of second order elliptic equation.

Note that the standard forms of the PDE is an expression with no mixed derivatives.

1.4.3 Reduction to Standard Form

1.4.4 Classification of Second Order Quasi-Linear

We aim now to generalise the classification of a two variable linear PDE to a n-variable quasi-linear PDE. Note that in the standard form with no mixed derivatives, the sign of the cofficients of highest order derivative are opposite for hyperbolic and same for elliptic. For parabolic, one of the coefficient of highest order vanishes. We shall use this approach to generalise the classification to multi-variable PDE. Consider the general second order quasi linear PDE with n independent variable

$$A(x, u, \nabla u) \cdot D^2 u + B(x, u) \cdot \nabla u = D(x),$$

where $A = A_{ij}$ is an $n \times n$ matrix with entries $A_{ij}(x, u, \nabla u)$, $D^2 u$ is the Hessian matrix and $B = (B_i)$. The first dot product in second order part (involving A) is in $\mathbb{R}^{n^2}$ and the second dot product, involving B, is in $\mathbb{R}^n$. We consider the above PDE at a specific point x_0. Let $u(x_0) = u_0$ and $A_0 = A(x_0, u_0, \nabla u(x_0))$. Thus, locally the PDE can be approximated by a constant coefficient equation whose highest order term involves the matrix multiplication $A_0 D^2 u$, where A_0 is a $n \times n$ matrix with constant entries. We assume the mixed derivatives to be equal and also assume A_0 is symmetric. In fact if A_0 is not symmetric, we can replace A_0 with $\frac{1}{2}(A_0 + A_0^t)$, which is symmetric. Following our argument in classifying Linear PDE in two variable, we look for the form of the highest order derivative under coordinate transformation. Let T be a $n \times n$ matrix with constant coefficients which transforms $w = Tx$. Applying,

$$\frac{\partial}{\partial x_i} \equiv \sum_{k=1}^{n} \frac{\partial w_k}{\partial x_i} \frac{\partial}{\partial w_k}$$

in $A_0 D_x^2 u$, we get $T A_0 T^t D_w^2 u$, where the subscript in the Hessian matrix denotes the variable in which the derivatives are taken and T^t denotes the transpose of the coordinate transformation matrix T.

We know from linear algebra that any real symmetric matrix can always be diagonalised. Since A_0 was assumed to be symmetric, we can choose the coordinate transformation T to be the one that diagonalises the matrix A_0. Thus, choosing T to be the diagonalising matrix of A_0, we get $T A_0 T^t$ as a diagonal matrix with diagonal entries, say $\lambda_1, \lambda_2, \ldots, \lambda_n$. Since A was real symmetric all $\lambda_i \in \mathbb{R}$, for all i. Thus, we classify PDE at a point x_0 based on the eigenvalues of the matrix A_0.

We say a PDE is *hyperbolic* at a point x_0, where the solution is u_0, if none of the eigenvalues λ_i of the coefficient matrix A_0 vanish and one eigenvalue has a sign opposite to others. We say it is *parabolic* if only one of λ_i vanishes and remaining have same sign. We say it is *elliptic*, if none of the eigenvalues λ_i vanish and all have same sign. For more that two independent variable, we may have other cases depending on the number of eigenvalues vanish and the pattern of sign of the remaining eigenvalues. But we do not deal with such equations in our context.

Chapter 2

First Order PDE

We begin by considering a simple linear first order evolution equation which will motivate the method of characteristics.

2.1 Linear Transport Equation

We consider the homogeneous initial-value transport problem. Given a vector $b \in \mathbb{R}^n$ and a function $g : \mathbb{R}^n \to \mathbb{R}$, we need to find $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ satisfying

$$\left\{ \begin{aligned} u_t(x,t) + b \cdot \nabla u(x,t) &= 0 && \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) &= g(x) && \text{on } \mathbb{R}^n \times \{t = 0\}. \end{aligned} \right. \tag{2.1.1}$$

By setting a new variable $y = (x,t)$ in $\mathbb{R}^n \times (0,\infty)$, (2.1.1) can be rewritten as

$$(b, 1) \cdot \nabla_y u(y) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty).$$

This means that the directional derivative of $u(y)$ along the direction $(b, 1)$ is zero. Thus, u must be constant along all lines in the direction of $(b, 1)$. The parametric representation of a line passing through a given point $(x,t) \in \mathbb{R}^n \times [0, \infty)$ and in the direction of $(b, 1)$ is given by $s \mapsto (x + sb, t + s)$, for all $s \geq -t$. Thus, u is constant on the line $(x + sb, t + s)$ for all $s \geq -t$ and, in particular, the value of u at $s = 0$ and $s = -t$ are same. Hence,

$$u(x,t) = u(x - tb, 0) = g(x - tb).$$

Since (x,t) was arbitrary in $\mathbb{R}^n \times (0, \infty)$, we have $u(x,t) = g(x - tb)$ for all $x \in \mathbb{R}^n$ and $t \geq 0$. The procedure explained above can be formalised as below.

The equation of a line passing through (x, t) and parallel to $(b, 1)$ is $(x, t)+ s(b, 1)$, for all $s \in (-t, \infty)$, i.e., $(x + sb, t + s)$. Thus, for a fixed $(x, t) \in \mathbb{R}^n \times (0, \infty)$, we set $v(s) := u(x+sb, t+s)$ for all $s \in (-t, \infty)$. Consequently,

$$\begin{aligned}\frac{dv(s)}{ds} &= \nabla u(x + sb, t + s) \cdot \frac{d(x + sb)}{ds} + \frac{\partial u}{\partial t}(x + sb, t + s)\frac{d(t + s)}{ds} \\ &= \nabla u(x + sb, t + s) \cdot b + \frac{\partial u}{\partial t}(x + sb, t + s)\end{aligned}$$

and from (2.1.1) we have $\frac{dv}{ds} = 0$ and hence $v(s) \equiv$ constant for all $s \in \mathbb{R}$. Thus, in particular, $v(0) = v(-t)$ which implies that $u(x, t) = u(x - tb, 0)$. But using the initial condition on u, we have $u(x, t) = g(x - tb)$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$. Thus, $g(x - tb)$ is a classical solution to (2.1.1) if $g \in C^1(\mathbb{R}^n)$. If $g \notin C^1(\mathbb{R}^n)$, we shall call $g(x - tb)$ to be a *weak* solution of (2.1.1).

The graph of $u(x, t)$ in $\mathbb{R}^{n+2}$ is a surface that looks like a wave propogating along the x-axis with velocity b, hence the name transport equation.

We now consider the inhomogeneous initial-value transport problem. Given a vector $b \in \mathbb{R}^n$, a function $g : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$, find $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ satisfying

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) + b \cdot \nabla u(x, t) = f(x, t) & \text{in } \mathbb{R}^n \times (0, \infty) \\ u(x, 0) = g(x) & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{2.1.2}$$

As before, we set $v(s) := u(x+sb, t+s)$ for all $s \in \mathbb{R}$ and for any given point $(x, t) \in \mathbb{R}^n \times (0, \infty)$. Thus,

$$\frac{dv(s)}{ds} = b \cdot \nabla u(x + sb, t + s) + \frac{\partial u}{\partial t}(x + sb, t + s) = f(x + sb, t + s).$$

Consider,

$$\begin{aligned}u(x, t) - g(x - tb) &= v(0) - v(-t) \\ &= \int_{-t}^{0} \frac{dv}{ds}\, ds \\ &= \int_{-t}^{0} f(x + sb, t + s)\, ds \\ &= \int_{0}^{t} f(x + (s - t)b, s)\, ds.\end{aligned}$$

Thus, $u(x,t) = g(x-tb) + \int_0^t f(x+(s-t)b, s)\, ds$ solves (2.1.2).

Observe that we derived the solution by converting the given PDE to a ODE. This technique is called the *method of characteristics* and can be applied to much more general equations. To motivate this method we re-derive the proof of transport equation. Recall that we solved the transport equation by finding a curve $(x(s), t(s))$ in $\mathbb{R}^n \times (0,\infty)$ intersecting the boundary and on which u is constant. Since s was dependent on t, we can as well look for a curve $x(t)$ in $\mathbb{R}^n$. Thus, we look for a curve $x(t)$ such that

$$\frac{d}{dt}u(x(t),t) = b \cdot \nabla u(x(t),t) + u_t(x(t),t).$$

Thus, we see that $\frac{dx_i(t)}{dt} = b_i$, for all $i = 1,2,\ldots,n$. Hence $x(t) = bt + x_0$, where $x_0 = x(0)$. Thus, $u(x_0) = u(x-tb) = g(x-tb)$.

2.2 Integral Surfaces and Monge Cone

Let us begin by considering a first order quasi-linear PDE,

$$F(\nabla u, u, x) := b(x,u(x)) \cdot \nabla u(x) - c(x,u(x)) = 0 \quad \text{for } x \in \Omega.$$

Thus, we have $(b(x,u(x)), c(x,u(x))) \cdot (\nabla u(x), -1) = 0$. Let S be a surface in $\mathbb{R}^{n+1}$ represented by the graph of the solution u of the given quasi-linear PDE. Let $(x,z) \in \mathbb{R}^n \times \mathbb{R}$. Thus,

$$S = \{(x,z) \in \Omega \times \mathbb{R} \mid u(x) - z = 0\}.$$

If we set $f(x,z) = u(x) - z$, then the normal to S is given by the gradient of f. Hence, $\nabla f = (\nabla u(x), -1)$. Therefore, for every point $(x_0, u(x_0)) \in S$, the coefficient vector $(b(x_0,u(x_0)), c(x_0,u(x_0))) \in \mathbb{R}^{n+1}$ is perpendicular to the normal vector $(\nabla u(x_0), -1)$. Thus, the coefficient vector must lie on the tangent plane at $(x_0, u(x_0))$ of S. Define the vector field $\mathbf{v}(x,z) = (b(x,z), c(x,z))$ formed by the coefficients of the quasi-linear PDE. Then, we note from the discussion above that S must be tangent at each of its point to the coefficient vector field $\mathbf{v}$.

Definition 2.2.1. *A curve in $\mathbb{R}^n$ is said to be an* integral curve *for a given vector field, if the curve is tangent to the vector field at each of its point. Similarly, a surface in $\mathbb{R}^n$ is said to be an* integral surface *for a given vector field, if the surface is tangent to the vector field at each of its point.*

In the spirit of the above definition and arguments, we note that S is an integral surface corresponding to the coefficient vector field $\mathbf{v}$. Thus, finding a solution to the quasi linear PDE is equivalent to finding a integral surface S corresponding to the coefficient vector field $\mathbf{v}$.

These arguments can be carried over to a general nonlinear first order PDE. Consider the first order nonlinear PDE, $F : \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ such that (cf. (1.3.1))

$$F(\nabla u(x), u(x), x) = 0 \quad \text{in } \Omega, \tag{2.2.1}$$

where $\Omega \subset \mathbb{R}^n$, F is given and u is the unknown to be found. Let S be the surface described by the graph of u in $\mathbb{R}^{n+1}$. For any fixed $(x, z) \in \Omega \times \mathbb{R}$, we can consider the equation $F(p, z, x) = 0$. If $(x, z) \in S$, then $z = u(x)$ and $p = \nabla u(x)$, since u solves (2.2.1). Thus, solving (2.2.1) is equivalent to finding a surface $S \in \mathbb{R}^{n+1}$ such that for all $x \in \Omega$, there is a $z \in \mathbb{R}$ such that $(x, z) \in S$ and $p = \nabla u(x)$.

Taking cue from the quasi-linear PDE, we expect (2.2.1) to be a relation between the points $(x, z) \in S$ and the normal vector $(p, -1)$ at each point of S. Let

$$V(x, z) := \{p \in \mathbb{R}^n \mid F(p, z, x) = 0\} \quad \text{for fixed } (x, z) \in \Omega \times \mathbb{R}.$$

Let S be the graph of u that solves (2.2.1) and let $(x_0, z_0) \in S$. As p varies in $V(x, z)$, we have a family of planes through (x_0, z_0) given by the equation

$$(z - z_0) = p \cdot (x - x_0).$$

The *envelope* of this family is a cone $C(x_0, z_0)$, called *Monge cone*, with vertex at (x_0, z_0). The envelope of the family of planes is that surface which is tangent at each of its point to some plane from the family.

Definition 2.2.2. *A surface S in $\mathbb{R}^{n+1}$ is said to be an* integral surface *if at each point $(x_0, z_0) \in S \subset \mathbb{R}^n \times \mathbb{R}$ it is tangential to the Monge cone with vertex at (x_0, z_0), i.e., there is a $p \in V(x_0, z_0)$ such that S is tangent to the plane corresponding to p.*

2.3 Cauchy Problem

In general the solution of a first order PDE is not unique. In other words, there may be many integral surfaces corresponding to a given PDE. Thus, solving a first order PDE should be more specific. A Cauchy problem states that: given a curve $\Gamma \subset \mathbb{R}^{n+1}$, can we find a solution u of (2.2.1) whose graph contains Γ.

2.4 Method of Characteristics

Solving a first order PDE is equivalent to finding an integral surface corresponding to the given PDE. The integral surfaces are usually the union of integral curves, also called the *characteristic* curves. Thus, finding an integral surface boils down to finding a family of characteristic curves. The method of characteristics gives the equation to find these curves in the form of a system of ODE.

Let us consider the first order PDE (2.2.1) in new independent variables $p \in \mathbb{R}^n$, $z \in \mathbb{R}$ and $x \in \Omega$. Consequently,

$$F(p, z, x) = F(p_1, p_2, \ldots, p_n, z, x_1, x_2, \ldots, x_n)$$

is a map of $2n + 1$ variable. We now introduce the derivatives (assume it exists) of F corresponding to each variable,

$$\begin{cases} \nabla_p F &= (F_{p_1}, \ldots, F_{p_n}) \\ \nabla_z F &= F_z \\ \nabla_x F &= (F_{x_1}, \ldots, F_{x_n}). \end{cases}$$

The method of characteristics reduces a given first order PDE to a system of ODE. The present idea is a generalisation of the idea employed in the study of linear transport equation (cf. (2.1.1)). We must choose a curve $x(s)$ in Ω such that we can compute u and ∇u along this curve. In fact, we would want the curve to intersect the boundary.

We begin by differentiating F w.r.t x_i in (2.2.1), we get

$$\sum_{j=1}^{n} F_{p_j} u_{x_j x_i} + F_z u_{x_i} + F_{x_i} = 0.$$

Thus, we seek to find $x(s)$ such that

$$\sum_{j=1}^{n} F_{p_j}(p(s), z(s), x(s)) u_{x_j x_i}(x(s)) + F_z(p(s), z(s), x(s)) p_i(s) + F_{x_i}(p(s), z(s), x(s)) = 0.$$

To free the above equation of second order derivatives, we differentiate $p_i(s)$ w.r.t s,

$$\frac{dp_i(s)}{ds} = \sum_{j=i}^{n} u_{x_i x_j}(x(s)) \frac{dx_j(s)}{ds}$$

and set

$$\frac{dx_j(s)}{ds} = F_{p_j}(p(s), z(s), x(s)).$$

Thus,

$$\frac{dx(s)}{ds} = \nabla_p F(p(s), z(s), x(s)). \tag{2.4.1}$$

Now substituting this in the first order equation, we get

$$\frac{dp_i(s)}{ds} = -F_z(p(s), z(s), x(s))p_i(s) - F_{x_i}(p(s), z(s), x(s)).$$

Thus,

$$\frac{dp(s)}{ds} = -\nabla_z F(p(s), z(s), x(s))p(s) - \nabla_x F(p(s), z(s), x(s)). \tag{2.4.2}$$

Similarly, we differentiate $z(s)$ w.r.t s,

$$\begin{aligned} \frac{dz(s)}{ds} &= \sum_{j=i}^{n} u_{x_j}(x(s)) \frac{dx_j(s)}{ds} \\ &= \sum_{j=i}^{n} u_{x_j}(x(s)) F_{p_j}(p(s), z(s), x(s)) \end{aligned}$$

Thus,

$$\frac{dz(s)}{ds} = p(s) \cdot \nabla_p F(p(s), z(s), x(s)). \tag{2.4.3}$$

We have $2n+1$ first order ODE called the *characteristic equations* of (2.2.1). The steps derived above can be summarised in the following theorem:

Theorem 2.4.1. *Let $u \in C^2(\Omega)$ solve* (2.2.1) *and $x(s)$ solve* (2.4.1)*, where $p(s) = \nabla u(x(s))$ and $z(s) = u(x(s))$. Then $p(s)$ and $z(s)$ solve* (2.4.2) *and* (2.4.3)*, respectively, for all $x(s) \in \Omega$.*

We end this section with remark that for linear, semi-linear and quasi-linear PDE one can do away with (2.4.2), the ODE corresponding to p because for these problems (2.4.3) and (2.4.1) form a determined system. However, for a fully nonlinear PDE one needs to solve all the 3 ODE's to compute the characteristic curve. The method of characteristics may be generalised to higher order hyperbolic PDE's.

2.4.1 Examples

We shall now illustrate the method of characteristics for various classes of first order PDE. We shall start with an example of linear problem. Let

$$F(\nabla u, u, x) := b(x) \cdot \nabla u(x) + c(x)u(x) = 0 \quad x \in \Omega.$$

Then, in the new variable, $F(p, z, x) = b(x) \cdot p + c(x)z$. Therefore, by (2.4.1), we have

$$\frac{dx(s)}{ds} = \nabla_p F = b(x(s)).$$

Also,

$$\frac{dz(s)}{ds} = b(x(s)) \cdot p(s) = b(x(s)) \cdot \nabla u(x(s)) = -c(x(s))z(s).$$

Example 1. Let $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0, x_2 > 0\}$. Let $\Gamma := \{(x_1, 0) \mid x_1 > 0\}$. Consider the linear PDE

$$\begin{cases} x_1 \frac{\partial u}{\partial x_2}(x_1, x_2) - x_2 \frac{\partial u}{\partial x_2}(x_1, x_2) &= u(x_1, x_2) \quad \text{in } \Omega \\ u(x_1, 0) &= g(x_1) \quad \text{on } \Gamma. \end{cases}$$

Thus, we have $b(x_1, x_2) = (-x_2, x_1)$ and $c(x_1, x_2) = -1$. Therefore, $\frac{dx_1(s)}{ds} = -x_2$ and $\frac{dx_2(s)}{ds} = x_1$. Hence, $\frac{d^2x_2(s)}{ds} = -x_2(s)$. Also, $\frac{dz(s)}{ds} = z(s)$. Thus, we have $(x_1(s), x_2(s)) = (x_1(0)\cos s, x_1(0)\sin s)$, where $x_1(0) > 0$ and $0 \leq s \leq \pi/2$. Also, $z(s) = z(0)e^s$. Note that if $s = 0$, we have $(x_1(0), x_2(0)) \in \Gamma$. Thus, $z(0) = u(x_1(0), x_2(0)) = g(x_1(0))$. Moreover, $x_1(0) = (x_1^2(s) + x_2^2(s))^{1/2}$, for all s and $s = \tan^{-1}(x_2(s)/x_1(s))$. Therefore, for any given (x_1, x_2), we have

$$u(x_1, x_2) = u(x_1(s), x_2(s)) = z(s) = g(x_1(0))e^s = g(\sqrt{x_1^2 + x_2^2})e^{\tan^{-1}(x_2/x_1)}.$$

□

A quasi-linear PDE has the form

$$F(\nabla u, u, x) := b(x, u(x)) \cdot \nabla u(x) + c(x, u(x)) = 0 \quad \text{for } x \in \Omega.$$

Then, in the new variable, $F(p, z, x) = b(x, z) \cdot p + c(x, z)$. Therefore, by (2.4.1), we have

$$\frac{dx(s)}{ds} = \nabla_p F = b(x(s), z(s)).$$

Also,

$$\frac{dz(s)}{ds} = b(x(s), z(s)) \cdot p(s) = b(x(s), z(s)) \cdot \nabla u(x(s)) = -c(x(s), z(s)).$$

Similarly, for a semi-linear PDE

$$F(\nabla u, u, x) := b(x) \cdot \nabla u(x) + c(x, u(x)) = 0 \quad \text{for } x \in \Omega,$$

we have

$$\frac{dx(s)}{ds} = \nabla_p F = b(x(s)).$$

Also,

$$\frac{dz(s)}{ds} = b(x(s), z(s)) \cdot p(s) = b(x(s), z(s)) \cdot \nabla u(x(s)) = -c(x(s), z(s)).$$

Example 2. Let $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$. Let $\Gamma := \{(x_1, 0) \mid x_1 \in \mathbb{R}\}$. Consider the semi-linear PDE

$$\begin{cases} \frac{\partial u}{\partial x_1}(x_1, x_2) + \frac{\partial u}{\partial x_2}(x_1, x_2) & = u^2(x_1, x_2) \quad \text{in } \Omega \\ u(x_1, 0) & = g(x_1) \quad \text{on } \Gamma. \end{cases}$$

Thus, we have $b(x, z) = (1, 1)$ and $c(x, z) = -z^2$. Therefore, $\frac{dx_1(s)}{ds} = 1$, $\frac{dx_2(s)}{ds} = 1$ and $\frac{dz(s)}{ds} = z^2(s)$. Thus, we have $(x_1(s), x_2(s)) = (s + x_1(0), s)$, where $s \geq 0$ and $x_1(0) \in \mathbb{R}$. Also, $z(s) = \frac{z(0)}{1-z(0)s}$. Note that if $s = 0$, we have $(x_1(0), 0) \in \Gamma$. Thus, $z(0) = u(x_1(0), 0) = g(x_1(0))$. Moreover, $x_1(0) = x_1(s) - x_2(s)$, for all s and $s = x_2(s)$. Therefore, $u(x_1, x_2) = u(x_1(s), x_2(s)) = z(s) = \frac{g(x_1(0))}{1-g(x_1(0))s} = \frac{g(x_1 - x2)}{1-g(x_1-x_2)x_2}$. □

Example 3. We now give an example of a fully non linear PDE. Consider Let $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0\}$. Let $\Gamma := \{(0, x_2) \mid x_2 \in \mathbb{R}\}$. Consider the fully non-linear PDE

$$\begin{cases} \frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2} & = u(x_1, x_2) \quad \text{in } \Omega \\ u(0, x_2) & = x_2^2 \quad \text{on } \Gamma. \end{cases}$$

We have $F(p, z, x) = p_1 p_2 - z$. Thus, using (2.4.1), we get

$$\frac{dx(s)}{ds} = (p_2(s), p_1(s)).$$

From (2.4.2), we get

$$\frac{dp(s)}{ds} = p(s)$$

and from (2.4.3) we get

$$\frac{dz(s)}{ds} = (p_1(s), p_2(s)) \cdot (p_2(s), p_1(s)) = 2p_1(s)p_2(s).$$

Integrating, we get $p(s) = p(0)e^s$, for all $s \in \mathbb{R}$. Using p, we solve for x to get $x_1(s) = p_2(0)(e^s - 1)$ and $x_2(s) = p_1(0)(e^s - 1) + x_2(0)$, where $x_2(0) \in \mathbb{R}$. Solving for z, we get

$$z(s) = p_1(0)p_2(0)(e^{2s} - 1) + z(0),$$

where $z(0) = x_2^2(0)$. Thus, determining $p_1(0), p_2(0)$, we solve for z, in turn u. Since $u(0, x_2) = x_2^2$ and $p_2 = \frac{\partial u}{\partial x_2}(0, x_2) = 2x_2$, we get $p_2(0) = 2x_2(0)$. Moreover, since u solves the non-linear PDE, we get

$$p_1(0)p_2(0) = z(0) = u(0, x_2(0)) = x_2^2(0).$$

Thus, $p_1(0) = x_2(0)/2$. Thus, $z(s) = x_2^2(0)e^{2s}$. Using the equation for x, we solve for s and $x_2(0)$. Thus,

$$x_2(0) = \frac{4x_2 - x_1}{4} \text{ and } e^s = \frac{x_1 + 4x_2}{4x_2 - x_1}.$$

Hence $u(x_1, x_2) = \frac{(x_1+4x_2)^2}{16}$. □

2.5 Complete Integrals and General Solutions

In this section, we introduce some simple classes of solutions and generate much general solutions from these for a general nonlinear first order PDE (2.2.1),i.e,

$$F(\nabla u, u, x) = 0 \quad x \in \Omega.$$

Let $A \subset \mathbb{R}^n$ be an open set which is the parameter set. Let us introduce the $n \times (n+1)$ matrix

$$(D_a u, D^2_{xa} u) := \begin{pmatrix} u_{a_1} & u_{x_1 a_1} & \dots & u_{x_n a_1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{a_n} & u_{x_1 a_n} & \dots & u_{x_n a_n} \end{pmatrix}.$$

Definition 2.5.1. *A C^2 function $u = u(x; a)$ is said to be a* complete integral *in $\Omega \times A$ if $u(x; a)$ solves* (2.2.1) *for each $a \in A$ and the rank of the matrix $(D_a u, D^2_{xa} u)$ is n.*

The condition on the rank of the matrix means that $u(x; a)$ strictly depends on all the n components of a and there is no C^1 map $\phi : A \to \mathbb{R}^{n-1}$ and a solution v of (2.2.1) such that $u(x; a) = v(x; \phi(a))$.

The graph of a complete integral S_a depends on the parameter a chosen. To choose the envelope of the surfaces S_a, we need to have $a = \phi(x)$. The envelope formed by the family of integral surfaces S_ϕ, the integral surface corresponding to $u(x, \phi(x))$, is the general solution.

Definition 2.5.2.

Chapter 3

Second Order PDE

3.1 The Laplacian

Recall that we introduced Laplacian to be the trace of the Hessain matrix, $\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. The Laplace operator usually appears in physical models associated with dissipative effects (except wave equation). The importance of Laplace operator can be realised by its appearance in various physical models. For instance, the heat equation

$$\frac{\partial}{\partial t} - \Delta,$$

the wave equation

$$\frac{\partial^2}{\partial t^2} - \Delta,$$

or the Schrödinger's equation

$$i\frac{\partial}{\partial t} + \Delta.$$

The Laplacian is a linear operator, i.e., $\Delta(u+v) = \Delta u + \Delta v$ and $\Delta(\lambda u) = \lambda \Delta u$ for any constant $\lambda \in \mathbb{R}$.

3.1.1 Laplacian in Different Coordinate Systems

As we know, in cartesian coordiantes, the n-dimensional Laplacian is given as

$$\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

In polar coordinates (2 dimensions), the Laplacian is given as

$$\Delta := \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

where r is the magnitude component and θ is the direction component. In cylindrical coordinates (3 dimensions), the Laplacian is given as

$$\Delta := \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

In spherical coordinates (3 dimensions), the Laplacian is given as

$$\Delta := \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial}{\partial \phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial \theta^2}.$$

3.1.2 Harmonic Functions

We already remarked that every scalar potential is a harmonic function. Gauss was the first to deduce some important properties of harmonic functions and thus laid the foundation for Potential theory and Harmonic Analysis.

Definition 3.1.1. *Let Ω be an open subset of $\mathbb{R}^n$. A function $u \in C^2(\Omega)$ is said to be* harmonic *on Ω if $\Delta u(x) = 0$ in Ω.*

Due to the linearity of Δ, sum of any finite number of harmonic functions is harmonic and a scalar multiple of a harmonic function is harmonic. Moreover, harmonic functions can be viewed as the kernel of the Laplace operator, say from C^2 to the space of continuous functions. It is trivial to check that all linear polynomials up to degree one are harmonic, i.e.,

$$\sum_{|\alpha|\leq 1} a_\alpha x^\alpha$$

are harmonic. But we also have functions of higher degree and functions not expressible in terms of elementary functions as harmonic functions. For instance, note that $u(x) = \prod_{i=1}^n x_i$ is harmonic. Similarly, for instance when $n = 2$, $u(x) = x_1^2 - x_2^2$, $u(x) = e^{x_1}\sin x_2$ or $u(x) = e_1^x \cos x_2$ are all harmonic. Our aim in this section is to understand the properties of harmonic functions.

In fact, we shall note later that any harmonic function is C^∞. Also, note that if u is a harmonic function on Ω then, by Gauss divergence theorem (cf. Theorem C.0.7,

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, d\sigma = 0.$$

Definition 3.1.2. *Let Ω be an open subset of $\mathbb{R}^n$ and $w_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ (cf. Appendix B) be the surface area of the unit sphere $S_1(0)$ of $\mathbb{R}^n$.*

(a) A function $u \in C(\Omega)$ is said to satisfy the first mean value property *(I-MVP) in Ω if*

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int_{S_r(x)} u(y)\, d\sigma_y \quad \textit{for any } B_r(x) \subset \Omega.$$

(b) A function $u \in C(\Omega)$ is said to satisfy the second mean value property *(II-MVP) in Ω if*

$$u(x) = \frac{n}{\omega_n r^n} \int_{B_r(x)} u(y)\, dy \quad \textit{for any } B_r(x) \subset \Omega.$$

Exercise 1. Show that u satisfies the I-MVP iff

$$u(x) = \frac{1}{\omega_n} \int_{S_1(0)} u(x + rz)\, d\sigma_z.$$

Similarly, u satisfies II-MVP iff

$$u(x) = \frac{n}{\omega_n} \int_{B_1(0)} u(x + rz)\, dz.$$

Exercise 2. Show that the first MVP and second MVP are equivalent. That is show that u satisfies (a) iff u satisfies (b).

Owing to the above exercise we shall, henceforth, refer to the I-MVP and II-MVP as just mean value property (MVP).

We shall now prove a result on the smoothness of a function satisfying MVP.

Theorem 3.1.3. *If $u \in C(\Omega)$ satisfies the MVP in Ω, then $u \in C^\infty(\Omega)$.*

Proof. We first consider $u_\varepsilon := \rho_\varepsilon * u$, the convolution of u with mollifiers, as introduced in Theorem D.0.9. where

$$\Omega_\varepsilon := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}.$$

We shall now show that $u = u_\varepsilon$ for all $\varepsilon > 0$, due to the MVP of u and the radial nature of ρ. Let $x \in \Omega_\varepsilon$. Consider

$$\begin{aligned}
u_\varepsilon(x) &= \int_\Omega \rho_\varepsilon(x-y)u(y)\,dy \\
&= \int_{B_\varepsilon(x)} \rho_\varepsilon(x-y)u(y)\,dy \quad \text{(Since supp}(\rho_\varepsilon)\text{ is in } B_\varepsilon(x)) \\
&= \int_0^\varepsilon \rho_\varepsilon(r)\left(\int_{S_r(x)} u(y)\,d\sigma_y\right) dr \quad \text{(cf. Theorem B.0.2)} \\
&= u(x)\omega_n \int_0^\varepsilon \rho_\varepsilon(r) r^{n-1}\,dr \quad \text{(Using MVP of } u) \\
&= u(x) \int_0^\varepsilon \rho_\varepsilon(r)\left(\int_{S_r(0)} d\sigma_y\right) dr \\
&= u(x) \int_{B_\varepsilon(0)} \rho_\varepsilon(y)\,dy = u(x).
\end{aligned}$$

Thus $u_\varepsilon(x) = u(x)$ for all $x \in \Omega_\varepsilon$ and for all $\varepsilon > 0$. Since $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$ for all $\varepsilon > 0$ (cf. Theorem D.0.9), we have $u \in C^\infty(\Omega_\varepsilon)$ for all $\varepsilon > 0$. □

Theorem 3.1.4. *Let u be a harmonic function on Ω. Then u satisfies the MVP in Ω.*

Proof. Let $B_r(x) \subset \Omega$ be any ball with centre at $x \in \Omega$ and for some $r > 0$. For the given harmonic function u, we set

$$v(r) := \frac{1}{\omega_n r^{n-1}} \int_{S_r(x)} u(y)\,d\sigma_y.$$

Note that v is not defined at 0, since $r > 0$. We have from Exercise 1 that

$$v(r) = \frac{1}{\omega_n} \int_{S_1(0)} u(x+rz)\,d\sigma_z.$$

Now, differentiating both sides w.r.t r, we get

$$\begin{aligned}\frac{dv(r)}{dr} &= \frac{1}{\omega_n}\int_{S_1(0)} \nabla u(x+rz)\cdot z\,d\sigma_z \\ &= \frac{1}{\omega_n r^{n-1}}\int_{S_r(x)} \nabla u(y)\cdot \frac{(y-x)}{r}\,d\sigma_y\end{aligned}$$

Since $|x-y|=r$, by setting $\nu := (y-x)/r$ as the unit vector, and applying the Gauss divergence theorem along with the fact that u is harmonic, we get

$$\frac{dv(r)}{dr} = \frac{1}{\omega_n r^{n-1}}\int_{S_r(x)} \nabla u(y)\cdot \nu\,d\sigma_y = \frac{1}{\omega_n r^{n-1}}\int_{B_r(x)} \Delta u(y)\,dy = 0.$$

Thus, v is a constant function of $r>0$ and hence

$$v(r) = v(\varepsilon) \quad \forall \varepsilon > 0.$$

Moreover, since v is continuous (constant function), we have

$$\begin{aligned}v(r) &= \lim_{\varepsilon\to 0} v(\varepsilon) \\ &= \lim_{\varepsilon\to 0}\frac{1}{\omega_n}\int_{S_1(0)} u(x+\varepsilon z)\,d\sigma_z \\ &= \frac{1}{\omega_n}\int_{S_1(0)} \lim_{\varepsilon\to 0} u(x+\varepsilon z)\,d\sigma_z \quad (u \text{ is continuous on } S_1(0)) \\ &= \frac{1}{\omega_n}\int_{S_1(0)} u(x)\,d\sigma_z \\ &= u(x) \quad (\text{Since } \omega_n \text{ is the surface area of } S_1(0)).\end{aligned}$$

Thus, u satisfies I-MVP and hence the II-MVP. □

Corollary 3.1.5. *If u is harmonic on Ω, then $u \in C^\infty(\Omega)$.*

The above corollary is a easy consequence of Theorem 3.1.4 and Theorem 3.1.3. We shall now prove that any function satisfying MVP is harmonic.

Theorem 3.1.6. *If $u \in C(\Omega)$ satisfies the MVP in Ω, then u is harmonic in Ω.*

Proof. Since u satisfies MVP, by Theorem 3.1.3, $u \in C^\infty(\Omega)$. Thus, Δu makes sense. Now, suppose u is not harmonic in Ω, then there is a $x \in \Omega$ such that $\Delta u(x) \neq 0$. Without loss of generality, let's say $\Delta u(x) > 0$. Moreover, since Δu is continuous there is a $s > 0$ such that, for all $y \in B_s(x)$, $\Delta u(y) > 0$. As done previously, we set for $r > 0$,

$$v(r) := \frac{1}{\omega_n r^{n-1}} \int_{S_r(x)} u(y)\, d\sigma_y.$$

Thus, $v(r) = u(x)$ for all $r > 0$ and hence v is a constant function of r and $v'(s) = 0$. But

$$0 = \frac{dv(s)}{dr} = \frac{1}{\omega_n r^{n-1}} \int_{B_s(x)} \Delta u(y)\, dy > 0$$

is a contradiction. Therefore, u is harmonic in Ω. □

Above results leads us to conclude that a function is harmonic iff it satisfies the MVP.

Exercise 3. If u_m is a sequence of harmonic functions in Ω converging to u uniformly on compact subsets of Ω, then show that u is harmonic in Ω.

Theorem 3.1.7 (Strong Maximum Principle)**.** *Let Ω be an open, connected (domain) subset of $\mathbb{R}^n$. Let u be harmonic in Ω and $M := \max_{y\in\overline{\Omega}} u(y)$. Then*

$$u(x) < M \quad \forall x \in \Omega$$

or $u \equiv M$ is constant in Ω.

Proof. We define a subset S of Ω as follows,

$$S := \{x \in \Omega \mid u(x) = M\}.$$

If $S = \emptyset$, we have $u(x) < M$ for all $x \in \Omega$. Suppose $S \neq \emptyset$. Then S is closed subset of Ω, since u is continuous. Now, for any $x \in S$, by MVP

$$u(x) = \frac{n}{\omega_n r^n} \int_{B_r(x)} u(y)\, dy \quad \text{for every } r \text{ such that } B_r(x) \subset \Omega.$$

Thus, we have

$$M = u(x) = \frac{n}{\omega_n r^n} \int_{B_r(x)} u(y)\, dy \leq M$$

Hence equality will hold above only when $u(y) = M$ for all $y \in B_r(x)$. Thus, we have shown that for any $x \in S$, we have $B_r(x) \subset S$. Therefore, S is open. Since Ω is connected, the only open and closed subsets are $\emptyset$ or Ω. Since S was assumed to be non-empty, we should have $S = \Omega$. Thus, $u \equiv M$ is constant in Ω. □

Corollary 3.1.8 (Weak maximum Principle)**.** *Let Ω be an open, bounded subset of $\mathbb{R}^n$. Let $u \in C(\overline{\Omega})$ be harmonic in Ω. Then*

$$\max_{y \in \overline{\Omega}} u(y) = \max_{y \in \partial\Omega} u(y).$$

Proof. Let $M := \max_{y \in \overline{\Omega}} u(y)$. If there is a $x \in \Omega$ such that $u(x) = M$, then $u \equiv M$ is constant on the connected component of Ω containing x. Thus, $u = M$ on the boundary of the connected component which is a part of $\partial\Omega$. □

Theorem 3.1.9 (Uniqueness of Harmonic Functions)**.** *Let Ω be an open, bounded subset of $\mathbb{R}^n$. Let $u_1, u_2 \in C(\overline{\Omega})$ be harmonic in Ω such that $u_1 = u_2$ on $\partial\Omega$, then $u_1 = u_2$ in Ω.*

Proof. Note that $u_1 - u_2$ is a harmonic function and hence, by weak maximum principle, should attain its maximum on $\partial\Omega$. But $u_1 - u_2 = 0$ on $\partial\Omega$. Thus $u_1 - u_2 \leq 0$ in Ω. Now, repeat the argument for $u_2 - u_1$, we get $u_2 - u_1 \leq 0$ in Ω. Thus, we get $u_1 - u_2 = 0$ in Ω. □

Theorem 3.1.10 (Estimates on derivatives)**.** *If u is harmonic in Ω, then*

$$|D^\alpha u(x)| \leq \frac{C_k}{r^{n+k}} \|u\|_{1,B_r(x)} \quad \forall B_r(x) \subset \Omega \text{ and each } |\alpha| = k$$

where the constants $C_0 = \frac{n}{\omega_n}$ and $C_k = C_0(2^{n+1}nk)^k$ for $k = 1, 2, \ldots$.

Proof. We prove the result by induction on k. Let $k = 0$. Since u is harmonic, by II-MVP we have, for any $B_r(x) \subset \Omega$,

$$\begin{aligned}
|u(x)| &= \frac{n}{\omega_n r^n} \left| \int_{B_r(x)} u(y)\, dy \right| \\
&\leq \frac{n}{\omega_n r^n} \int_{B_r(x)} |u(y)|\, dy \\
&= \frac{n}{\omega_n r^n} \|u\|_{1,B_r(x)} = \frac{C_0}{r^n} \|u\|_{1,B_r(x)}.
\end{aligned}$$

Now, let $k = 1$. Observe that if u is harmonic then by differentiating the Laplace equation and using the equality of mixed derivatives, we have $u_{x_i} := \frac{\partial u}{\partial x_i}$ is harmoic, for all $i = 1, 2, \ldots, n$. Now, by the II-MVP of u_{x_i}, we have

$$\begin{aligned}
|u_{x_i}(x)| &= \frac{n2^n}{\omega_n r^n} \left| \int_{B_{r/2}(x)} u_{x_i}(y)\, dy \right| \\
&= \frac{n2^n}{\omega_n r^n} \left| \int_{S_{r/2}(x)} u\nu_i \, d\sigma_y \right| \quad \text{(by Gauss-Green theorem)} \\
&\leq \frac{2n}{r} \|u\|_{\infty, S_{r/2}(x)}.
\end{aligned}$$

Thus, it now remains to estimate $\|u\|_{\infty,S_{r/2}(x)}$. Let $z \in S_{r/2}(x)$, then $B_{r/2}(z) \subset B_r(x) \subset \Omega$. But, using $k = 0$ result, we have

$$|u(z)| \leq \frac{C_0 2^n}{r^n} \|u\|_{1,B_{r/2}(z)} \leq \frac{C_0 2^n}{r^n} \|u\|_{1,B_r(x)}.$$

Therefore, $\|u\|_{\infty,S_{r/2}(x)} \leq \frac{C_0 2^n}{r^n} \|u\|_{1,B_r(x)}$ and using this in the estimate of u_{x_i}, we get

$$|u_{x_i}(x)| \leq \frac{C_0 n 2^{n+1}}{r^{n+1}} \|u\|_{1,B_r(x)}.$$

Hence

$$|D^\alpha u(x)| \leq \frac{C_1}{r^{n+1}} \|u\|_{1,B_r(x)} \quad \text{for } |\alpha| = 1.$$

Let now $k \geq 2$ and α be a multi-index such that $|\alpha| = k$. We assume the induction hypothesis that the estimate to be proved is true for $k - 1$. Note that $D^\alpha u = \frac{\partial D^\beta u}{\partial x_i}$ for some $i \in \{1, 2, \ldots, n\}$ and $|\beta| = k - 1$. Moreover, if u is harmonic then by differentiating the Laplace equation and using the equality of mixed derivatives, we have $\frac{\partial D^\beta u}{\partial x_i}$ is harmoic for $i = 1, 2, \ldots, n$. Thus, following an earlier argument, we have

$$\begin{aligned}
|D^\alpha u(x)| = \left| \frac{\partial D^\beta u(x)}{\partial x_i} \right| &= \frac{nk^n}{\omega_n r^n} \left| \int_{B_{r/k}(x)} \frac{\partial D^\beta u(y)}{\partial x_i}\, dy \right| \\
&= \frac{nk^n}{\omega_n r^n} \left| \int_{S_{r/k}(x)} D^\beta u \nu_i \, d\sigma_y \right| \quad \text{(by Gauss-Green theorem)} \\
&\leq \frac{nk}{r} \|D^\beta u\|_{\infty, S_{r/k}(x)}.
\end{aligned}$$

It now only remains to estimate $\|D^\beta u\|_{\infty,S_{r/k}(x)}$. Let $z \in S_{r/k}(x)$, then $B_{(k-1)r/k}(z) \subset B_r(x) \subset \Omega$. But, using induction hypothesis for $k-1$, we have

$$|D^\beta u(z)| \leq \frac{C_{k-1}k^{n+k-1}}{((k-1)r)^{n+k-1}}\|u\|_{1,B_{(k-1)r/k}(z)} \leq \frac{C_{k-1}k^{n+k-1}}{((k-1)r)^{n+k-1}}\|u\|_{1,B_r(x)}.$$

Therefore, using the above estimate for $D^\alpha u$, we get

$$\begin{aligned}
|D^\alpha u(x)| &\leq \frac{C_{k-1}nk^{n+k}}{(k-1)^{n+k-1}r^{n+k}}\|u\|_{1,B_r(x)} \\
&= \frac{C_0 2^{(n+1)(k-1)}n^k(k-1)^{k-1}k^{n+k}}{(k-1)^{n+k-1}r^{n+k}}\|u\|_{1,B_r(x)} \\
&= \frac{C_0(2^{n+1}nk)^k}{r^{n+k}}\left(\frac{k}{k-1}\right)^n\left(\frac{1}{2^{n+1}}\right)\|u\|_{1,B_r(x)} \\
&= \frac{C_0(2^{n+1}nk)^k}{r^{n+k}}\left(\frac{k}{2(k-1)}\right)^n\left(\frac{1}{2}\right)\|u\|_{1,B_r(x)} \\
&\leq \frac{C_k}{r^{n+k}}\|u\|_{1,B_r(x)} \quad \text{since } \left(\frac{k}{2(k-1)}\right)^n\left(\frac{1}{2}\right) \leq 1.
\end{aligned}$$

Hence

$$|D^\alpha u(x)| \leq \frac{C_k}{r^{n+k}}\|u\|_{1,B_r(x)} \quad \text{for } |\alpha| = k, \forall k \geq 2.$$

□

Theorem 3.1.11 (Liouville's Theorem)**.** *If u is bounded and harmonic on $\mathbb{R}^n$, then u is constant.*

Proof. For any $x \in \mathbb{R}^n$ and $r > 0$, we have the estimate on the first derivative as,

$$\begin{aligned}
|\nabla u(x)| &\leq \frac{C_1}{r^{n+1}}\|u\|_{1,B_r(x)} \\
&= \frac{2^{n+1}n}{\omega_n r^{n+1}}\|u\|_{1,B_r(x)} \\
&\leq \frac{2^{n+1}n}{\omega_n r^{n+1}}\|u\|_{\infty,\mathbb{R}^n}\omega_n r^n \\
&= \frac{2^{n+1}n}{r}\|u\|_{\infty,\mathbb{R}^n} \to 0 \text{ as } r \to \infty.
\end{aligned}$$

Thus, $\nabla u \equiv 0$ in $\mathbb{R}^n$ and hence u is constant. □

Exercise 4. Show that if u is harmonic in Ω, then u is analytic in Ω. (Hint: Use the estimates on derivatives with Stirling's formula and Taylor expansion).

We end our discussion on the properties of harmonic function with Harnack inequality. The Harnack inequality states that non-negative harmonic functions cannot be very large or very small at any point without being so everywhere in a compact set containing that point.

Theorem 3.1.12 (Harnack's Inequality)**.** *Let u be harmonic in Ω and $u \geq 0$ in Ω, then for each connected open subset $\omega \subset\subset \Omega$ there is a constant $C > 0$ (depending only on ω) such that*

$$\sup_{x\in\omega} u(x) \leq C \inf_{x\in\omega} u(x).$$

In particular,

$$\frac{1}{C}u(y) \leq u(x) \leq Cu(y) \quad \forall x, y \in \omega.$$

Proof. Set $r := \frac{1}{4}\text{dist}(\omega, \partial\Omega)$. Let $x, y \in \omega$ such that $|x-y| < r$. By II-MVP,

$$\begin{aligned} u(x) &= \frac{n}{\omega_n 2^n r^n} \int_{B_{2r}(x)} u(z)\, dz \\ &\geq \frac{n}{\omega_n 2^n r^n} \int_{B_r(y)} u(z)\, dz = \frac{1}{2^n} u(y). \end{aligned}$$

Thus, $1/2^n u(y) \leq u(x)$. Interchanging the role of x and y, we get $1/2^n u(x) \leq u(y)$. Thus, $1/2^n u(y) \leq u(x) \leq 2^n u(y)$ for all $x, y \in \omega$ such that $|x - y| \leq r$.

Now, let $x, y \in \omega$. Since $\overline{\omega}$ is compact and connected in Ω, we can pick points $x = x_1, x_2, \ldots, x_m = y$ such that $\cup_{i=1}^m B_i \supset \overline{\omega}$, where $B_i := B_{r/2}(x_i)$ and are sorted such that $B_i \cap B_{i+1} \neq \emptyset$, for $i = 1, 2, \ldots, m-1$. Hence, note that $|x_{i+1} - x_i| \leq r$. Therefore,

$$u(x) = u(x_1) \geq \frac{1}{2^n}u(x_2) \geq \frac{1}{2^{2n}}u(x_3) \geq \ldots \geq \frac{1}{2^{(m-1)n}}u(x_m) = \frac{1}{2^{(m-1)n}}u(y).$$

Thus, C can be chosen to be $\frac{1}{2^{(m-1)n}}$. □

Now that we have sufficient understanding of harmonic functions, solution of homogeneous Laplace equation, the natural question to ask is about the solution of Poisson equation, the inhomogeneous Laplace equation.

3.2 Poisson Equation in $\mathbb{R}^n$

We now wish to solve the Poisson equation, for any given f (under some hypothesis) find u such that

$$-\Delta u = f \text{ in } \mathbb{R}^n. \tag{3.2.1}$$

Recall that we have already introduced the notion convolution of functions (cf. Appendix D) while discussing C^∞ properties of harmonic functions. We also observed that the differential operator can be accumulated either side in the convolution operation. Suppose there is a "function" K with the property that ΔK is the identity of the convolution operation, i.e., $f * \Delta K = f$, then we know that $u := f * K$ is a solution of (3.2.1).

Definition 3.2.1. *We shall say a "function" K to be the* fundamental solution *of the Laplacian, Δ, if ΔK is the identity with respect to the convolution operation.*

We caution that the above definition is not mathematically precise because we made no mention on what the "function" K could be and its differentiability, even its existence is under question. We shall just take it as a informal definition.

The question of interest is: can one find a K with above property? To answer this, let us observe that since we want K such that $f * \Delta K$ for all f in the given space of functions in $\mathbb{R}^n$. In particular, one can choose $f \equiv 1$. Thus, we are looking for K such that $1 * \Delta K = 1$, i.e.,

$$\int_{\mathbb{R}^n} \Delta K = 1.$$

Equivalently, we want K such that

$$\lim_{r\to\infty} \int_{B_r(0)} \Delta K = 1,$$

which by Gauss divergence theorem (all informally) means we look for K such that

$$\lim_{r\to\infty} \int_{S_r(0)} \nabla K \cdot \nu = 1.$$

3.2.1 Fundamental Solution of Laplacian

The Laplace operator is invariant under coordinate translation and rotation. Suppose $x \in \mathbb{R}^n$ is translated by a vector $a \in \mathbb{R}^n$. Then the Laplace operator is invariant of translation. That is, if $\Delta u(x) = f(x)$, then $\Delta u(x + a) = f(x+a)$. Similarly, for any rotation, the Laplace operator remains unchanged, i.e., if $\Delta u(r, \theta) = f(r, \theta)$, then $\Delta u(r, \theta + \eta) = f(r, \theta + \eta)$.

Exercise 5. If T_a is a translation of a vector by a, i.e., $T_a(x) = x + a$ and if $\Delta u(x) = f(x)$, then show that $\Delta u(T_a(x)) = f(T_a(x))$. Also, if O is a $n \times n$ orthogonal matrix and if $\Delta u(x) = f(x)$, then show that $\Delta u(Ox) = f(Ox)$.

Thus, we say the Laplacian commutes with translation and rotation. A radial function is constant on every sphere about the origin. Since Laplacian commutes with rotations, it should map the class of all radial functions to itself. The invariance of Laplacian under rotation motivates us to look for a radial fundamental solution. To do so, we shall understand how Laplacian treats radial functions.

Proposition 3.2.2. *If u is a radial function, i.e., $u(x) = u(r)$ where $x \in \mathbb{R}^n$ and $r = |x|$, then*

$$\Delta u(x) = \frac{d^2u(r)}{dr^2} + \frac{(n-1)}{r}\frac{du(r)}{dr}.$$

Proof. Note that

$$\begin{aligned}
\frac{\partial r}{\partial x_i} = \frac{\partial |x|}{\partial x_i} &= \frac{\partial(\sqrt{x_1^2 + \ldots + x_n^2})}{\partial x_i} \\
&= \frac{1}{2}(x_1^2 + \ldots + x_n^2)^{-1/2}(2x_i) \\
&= \frac{x_i}{r}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
\Delta u(x) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial u(x)}{\partial x_i}\right) &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{du(r)}{dr}\frac{x_i}{r}\right) \\
&= \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\left(\frac{1}{r}\frac{du(r)}{dr}\right) + \frac{n}{r}\frac{du(r)}{dr} \\
&= \sum_{i=1}^{n} \frac{x_i^2}{r}\frac{d}{dr}\left(\frac{du(r)}{dr}\frac{1}{r}\right) + \frac{n}{r}\frac{du(r)}{dr} \\
&= \sum_{i=1}^{n} \frac{x_i^2}{r}\left\{\frac{1}{r}\frac{d^2u(r)}{dr^2} - \frac{1}{r^2}\frac{du(r)}{dr}\right\} + \frac{n}{r}\frac{du(r)}{dr} \\
&= \frac{r^2}{r}\left\{\frac{1}{r}\frac{d^2u(r)}{dr^2} - \frac{1}{r^2}\frac{du(r)}{dr}\right\} + \frac{n}{r}\frac{du(r)}{dr} \\
&= \frac{d^2u(r)}{dr^2} - \frac{1}{r}\frac{du(r)}{dr} + \frac{n}{r}\frac{du(r)}{dr} \\
&= \frac{d^2u(r)}{dr^2} + \frac{(n-1)}{r}\frac{du(r)}{dr}.
\end{aligned}
$$

Hence the result proved. □

Corollary 3.2.3. *If u is a radial function, i.e., $u(x) = u(r)$ on $\mathbb{R}^n$ then $\Delta u = 0$ on $\mathbb{R}^n \setminus \{0\}$ iff $u(r) = a + \frac{b}{2-n}r^{2-n}$, for $n \neq 2$ or $u(r) = a + b\log r$ for $n = 2$, where a, b are some constants.*

Proof. Observe that $\Delta u = 0$ iff $u''(r) + \frac{(n-1)}{r}u'(r) = 0$. Now, integrating both sides w.r.t r, we get

$$
\begin{aligned}
\frac{u''(r)}{u'(r)} &= \frac{(1-n)}{r} \\
\log u'(r) &= (1-n)\log r + \log b \\
u'(r) &= br^{(1-n)}
\end{aligned}
$$

If $n = 2$ then, integration both sides once again yields,

$$u(r) = b\log r + a.$$

But, if $n \neq 2$, then we have,

$$u(r) = \frac{b}{2-n}r^{2-n} + a.$$

□

The reason to choose the domain of the Laplacian as $\mathbb{R}^n \setminus \{0\}$ is because the operator involves a 'r' in the denominator. However, for one dimensional case we can let zero to be on the domain of Laplacian, since for $n = 1$, the Laplace operator is unchanged. Thus, for $n = 1$, $u(x) = a + bx$ is a harmonic function in $\mathbb{R}^n$.

Note that as $r \to 0$, $u(r) \to \infty$. Thus, u has a singularity at 0. In fact, for any given $w \in \mathbb{R}^n$, $\Delta u(x - w) = 0$ for all $x \in \mathbb{R}^n \setminus \{w\}$. By special choice of constants a, b, we shall define the radial fundamental solution. We shall choose a, b such that for every sphere $S_r(0)$ about the origin, we have

$$\int_{S_r(0)} \frac{d}{dr} u(r)\, d\sigma = 1.$$

Thus,

$$\begin{aligned} 1 = \int_{S_r(0)} \frac{d}{dr} u(r)\, d\sigma &= \frac{b}{2-n} \frac{d}{dr} r^{2-n} \int_{S_r(0)} d\sigma \\ &= b r^{1-n} r^{n-1} \omega_n. \end{aligned}$$

Thus, we choose $b = \frac{1}{\omega_n}$ and $a \equiv 0$, for $n \neq 2$ and for $n = 2$, we choose $a \equiv 0$ and $b = \frac{1}{2\pi}$. For convention sake, we shall add minus ("$-$") sign (notice the minus sign in (3.2.1)).

Definition 3.2.4. *We say $K(x)$, defined as*

$$K(x) := \begin{cases} -\frac{1}{2\pi} \log |x| & (n = 2) \\ \frac{|x|^{2-n}}{\omega_n (n-2)} & (n \geq 3), \end{cases}$$

is the fundamental solution *of the Laplacian.*

We end this section by emphasising that the notion of fundamental solution has a precise definition in terms of the Dirac measure. The Dirac measure, at a point $x \in \mathbb{R}^n$, is defined as,

$$\delta_x(E) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

for all measurable subsets E of the measure space $\mathbb{R}^n$. The Dirac measure has the property that

$$\int_E d\delta_x = 1$$

if $x \in E$ and zero if $x \notin E$. Also, for any integrable function f,

$$\int_{\mathbb{R}^n} f(y)\, d\delta_x = f(x).$$

In this new set-up a fundamental solution K can be defined as the solution corresponding to δ_x, i.e.,

$$-\Delta K = \delta_x \text{ in } \mathbb{R}^n.$$

Note that the above equation, as such, makes no sense because the RHS is a set-function taking subsets of $\mathbb{R}^n$ as arguments, whereas K is a function on $\mathbb{R}^n$. To give meaning to above equation, one needs to view δ_x as a distribution (introduced by L. Schwartz) and the equation should be interpreted in the distributional derivative sense. The Dirac measure is the distributional limit of the sequence of mollifiers, ρ_ε, in the space of distributions.

3.2.2 Solving Poisson Equation

In this section, we shall give a formula for the solution of the Poisson equation (3.2.1) in $\mathbb{R}^n$ in terms of the fundamental solution.

Theorem 3.2.5. *For any given $f \in C_c^2(\mathbb{R}^n)$, $u := K * f$ is a solution to the Poisson equation* (3.2.1).

Proof. By the property of convolution (cf. proof of Theorem D.0.9), we know that $D^\alpha u(x) = (K * D^\alpha f)(x)$ for all $|\alpha| \leq 2$. Since $f \in C_c^2(\mathbb{R}^n)$, we have $u \in C^2(\mathbb{R}^n$. The difficulty arises due to the singularity of K at the origin.

Consider, for any fixed $m > 0$,

$$\begin{aligned}
\Delta u(x) &= \int_{\mathbb{R}^n} K(y)\Delta_x f(x-y)\,dy \\
&= \int_{B_m(0)} K(y)\Delta_x f(x-y)\,dy + \int_{\mathbb{R}^n\setminus B_m(0)} K(y)\Delta_x f(x-y)\,dy \\
&= \int_{B_m(0)} K(y)\Delta_x f(x-y)\,dy + \int_{\mathbb{R}^n\setminus B_m(0)} K(y)\Delta_y f(x-y)\,dy \\
&= \int_{B_m(0)} K(y)\Delta_x f(x-y)\,dy + \int_{S_m(0)} K(y)\nabla_y f(x-y)\cdot\nu\,d\sigma_y \\
&\quad -\int_{\mathbb{R}^n\setminus B_m(0)} \nabla_y K(y)\cdot\nabla_y f(x-y)\,dy \quad \text{(cf. Corollary C.0.8)} \\
&= \int_{B_m(0)} K(y)\Delta_x f(x-y)\,dy + \int_{S_m(0)} K(y)\nabla_y f(x-y)\cdot\nu\,d\sigma_y \\
&\quad +\int_{\mathbb{R}^n\setminus B_m(0)} \Delta_y K(y) f(x-y)\,dy - \int_{S_m(0)} f(x-y)\nabla_y K(y)\cdot\nu\,d\sigma_y \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(cf. Corollary C.0.8)} \\
&= \int_{B_m(0)} K(y)\Delta_x f(x-y)\,dy + \int_{S_m(0)} K(y)\nabla_y f(x-y)\cdot\nu\,d\sigma_y \\
&\quad -\int_{S_m(0)} f(x-y)\nabla_y K(y)\cdot\nu\,d\sigma_y \\
&:= I_m(x) + J_m(x) + K_m(x).
\end{aligned}$$

But, due to the compact support of f, we have

$$|I_m(x)| \le \|D^2 f\|_{\infty,\mathbb{R}^n} \int_{B_m(0)} |K(y)|\,dy.$$

Thus, for $n = 2$,

$$|I_m(x)| \le \frac{m^2}{2}\left(\frac{1}{2} + |\log m|\right)\|D^2 f\|_{\infty,\mathbb{R}^n}$$

and for $n \ge 3$, we have

$$|I_m(x)| \le \frac{m^2}{2(n-2)}\|D^2 f\|_{\infty,\mathbb{R}^n}.$$

Hence, as $m \to 0$, $|I_m(x)| \to 0$. Similarly,

$$\begin{aligned} |J_m(x)| &\leq \int_{S_m(0)} |K(y)\nabla_y f(x-y)\cdot\nu|\,d\sigma_y \\ &\leq \|\nabla f\|_{\infty,\mathbb{R}^n}\int_{S_m(0)} |K(y)|\,d\sigma_y. \end{aligned}$$

Thus, for $n = 2$,

$$|J_m(x)| \leq m|\log m|\|\nabla f\|_{\infty,\mathbb{R}^n}$$

and for $n \geq 3$, we have

$$|J_m(x)| \leq \frac{m}{(n-2)}\|\nabla f\|_{\infty,\mathbb{R}^n}.$$

Hence, as $m \to 0$, $|J_m(x)| \to 0$. Now, to tackle the last term $K_m(x)$, we note that a simple computation yields that $\nabla_y K(y) = \frac{-1}{\omega_n|y|^n}y$. Since we are in the m radius sphere $|y| = m$. Also the unit vector ν outside of $S_m(0)$, as a boundary of $\mathbb{R}^n \setminus B_m(0)$, is given by $-y/|y| = -y/m$. Therefore,

$$\nabla_y K(y)\cdot\nu = \frac{1}{\omega_n m^{n+1}}y\cdot y = \frac{1}{\omega_n m^{n-1}}.$$

$$\begin{aligned} K_m(x) &= -\int_{S_m(0)} f(x-y)\nabla_y K(y)\cdot\nu\,d\sigma_y \\ &= \frac{-1}{\omega_n m^{n-1}}\int_{S_m(0)} f(x-y)\,d\sigma_y \\ &= \frac{-1}{\omega_n m^{n-1}}\int_{S_m(x)} f(y)\,d\sigma_y \end{aligned}$$

Since f is continuous, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. When $m \to 0$, we can choose m such that $m < \delta$ and for this m, we see that Now, consider

$$\begin{aligned} |K_m(x) - (-f(x))| &= \left|f(x) - \frac{1}{\omega_n m^{n-1}}\int_{S_m(x)} f(y)\,d\sigma_y\right| \\ &= \frac{1}{\omega_n m^{n-1}}\int_{S_m(x)} |f(x) - f(y)|\,d\sigma_y < \varepsilon. \end{aligned}$$

Thus, as $m \to 0$, $K_m(x) \to -f(x)$. Hence, u solves (3.2.1). □

Remark 3.2.6. Notice that in the proof above, we have used the Green's identity eventhough our domain is not bounded (which is a hypothesis for Green's identity). This can be justified by taking a ball bigger that $B_m(0)$ and working in the annular region, and later letting the bigger ball approach all of $\mathbb{R}^n$.

A natural question at this juncture is: Is every solution of the Poisson equation (3.2.1) of the form $K * f$. We answer this question in the following theorem.

Theorem 3.2.7. *Let $f \in C_c^2(\mathbb{R}^n)$ and $n \geq 3$. If u is a solution of* (3.2.1) *and u is bounded, then u has the form $u(x) = (K*f)(x)+C$, for any $x \in \mathbb{R}^n$, where C is some constant.*

Proof. We know that (cf. Theorem 3.2.5) $u'(x) := (K * f)(x)$ solves (3.2.1), the Poisson equation in $\mathbb{R}^n$. Moreover, u' is bounded for $n \geq 3$, since $K(x) \to 0$ as $|x| \to \infty$ and f has compact support in $\mathbb{R}^n$. Also, since u is given to be a bounded solution of (3.2.1), $v := u - u'$ is a bounded harmonic function. Hence, by Liouville's theorem, v is constant. Therefore $u = u' + C$, for some constant C. □

3.3 Dirichlet Problem

We turn our attention to studying Poisson equation in proper subsets of $\mathbb{R}^n$. Let Ω be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary $\partial\Omega$.

The Dirichlet problem is stated as follows: Given $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$, find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ \quad u &= g \quad \text{on } \partial\Omega \end{cases} \tag{3.3.1}$$

Thus, we call the boundary condition imposed above to be the Dirichlet boundary condition. Note that, by Theorem 3.1.9, the solution of Dirichlet problem, if it exists, is unique.

To begin with we shall focus on the study of Dirichlet problem. The Dirichlet problem can be solved in four different approaches, viz., using Green's function, Dirichlet principle, layer potentials and L^2-estimates. The last two approaches also solve Neumann problem.

3.3.1 Green's Function

To begin we shall motivate the derivation of Green's function. For any $x \in \Omega$, choose $m > 0$ such that $B_m(x) \subset \Omega$. Set $\Omega_m := \Omega \setminus B_m(x)$. Now applying the second identity of Corollary C.0.8 for any $u \in C^2(\overline{\Omega})$ and $v(y) = K(y - x)$, the fundamental solution on $\mathbb{R}^n \setminus \{x\}$, on the domain Ω_m, we get

$$
\begin{aligned}
\int_{\partial\Omega_m} \left(u(y)\frac{\partial K}{\partial \nu}(y-x) - K(y-x)\frac{\partial u(y)}{\partial \nu} \right) d\sigma_y &= \int_{\Omega_m} u(y)\Delta_y K(y-x)\, dy \\
&\quad - \int_{\Omega_m} K(y-x)\Delta_y u(y)\, dy \\
\int_{\partial\Omega_m} \left(u(y)\frac{\partial K}{\partial \nu}(y-x) - K(y-x)\frac{\partial u(y)}{\partial \nu} \right) d\sigma_y &= -\int_{\Omega_m} K(y-x)\Delta_y u(y)\, dy \\
\int_{\partial\Omega} \left(u(y)\frac{\partial K}{\partial \nu}(y-x) - K(y-x)\frac{\partial u(y)}{\partial \nu} \right) d\sigma_y & \\
+ \int_{\Omega} K(y-x)\Delta_y u(y)\, dy &= -\int_{S_m(x)} u(y)\frac{\partial K}{\partial \nu}(y-x)\, d\sigma_y \\
&\quad + \int_{S_m(x)} K(y-x)\frac{\partial u(y)}{\partial \nu}\, d\sigma_y \\
&\quad + \int_{B_m(x)} K(y-x)\Delta_y u(y)\, dy \\
&= K_m(x) + J_m(x) + I_m(x)
\end{aligned}
$$

The RHS is handled exactly as in the proof of Theorem 3.2.5, since u is a continuous function on the compact set $\overline{\Omega}$ and is bounded. We repeat the arguments below for completeness sake.

Consider the first term $K_m(x)$. Note that $\nabla_y K(y - x) = \frac{-1}{\omega_n |y-x|^n}(y - x)$. Since we are in the m radius sphere $|y - x| = m$. Also the unit vector ν inside of $S_m(x)$, as a boundary of $\Omega \setminus B_m(x)$, is given by $-(y - x)/|y - x| = -(y - x)/m$. Therefore,

$$
\nabla_y K(y - x) \cdot \nu = \frac{1}{\omega_n m^{n+1}}(y - x) \cdot (y - x) = \frac{1}{\omega_n m^{n-1}}.
$$

Thus,

$$\begin{aligned} K_m(x) &= -\int_{S_m(x)} u(y)\nabla_y K(y-x)\cdot \nu \, d\sigma_y \\ &= \frac{-1}{\omega_n m^{n-1}} \int_{S_m(x)} u(y)\, d\sigma_y \end{aligned}$$

Since u is continuous, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|u(x)-u(y)| < \varepsilon$ whenever $|x-y| < \delta$. When $m \to 0$, we can choose m such that $m < \delta$ and for this m, we see that Now, consider

$$\begin{aligned} |K_m(x) - (-u(x))| &= \left| u(x) - \frac{1}{\omega_n m^{n-1}} \int_{S_m(x)} u(y)\, d\sigma_y \right| \\ &= \frac{1}{\omega_n m^{n-1}} \int_{S_m(x)} |u(x) - u(y)|\, d\sigma_y < \varepsilon. \end{aligned}$$

Thus, as $m \to 0$, $K_m(x) \to -u(x)$.

We next consider the term $J_m(x)$,

$$\begin{aligned} |J_m(x)| &\leq \int_{S_m(x)} |K(y-x)\nabla_y u(y)\cdot \nu|\, d\sigma_y \\ &\leq \|\nabla_y u\|_{\infty,\Omega} \int_{S_m(x)} |K(y-x)|\, d\sigma_y. \end{aligned}$$

Thus, for $n = 2$,

$$|J_m(x)| \leq m|\log m| \|\nabla_y u\|_{\infty,\Omega}$$

and for $n \geq 3$, we have

$$|J_m(x)| \leq \frac{m}{(n-2)} \|\nabla_y u\|_{\infty,\Omega}.$$

Hence, as $m \to 0$, $|J_m(x)| \to 0$. We now consider the term I_m.

$$|I_m(x)| \leq \|D^2 u\|_{\infty,\Omega} \int_{B_m(x)} |K(y-x)|\, dy.$$

Thus, for $n = 2$,

$$|I_m(x)| \leq \frac{m^2}{2}\left(\frac{1}{2} + |\log m|\right) \|D^2 u\|_{\infty,\Omega}$$

and for $n \geq 3$, we have

$$|I_m(x)| \leq \frac{m^2}{2(n-2)} \|D^2 u\|_{\infty,\Omega}.$$

Hence, as $m \to 0$, $|I_m(x)| \to 0$. Therefore, letting $m \to 0$, we have the identity

$$u(x) = \int_{\partial\Omega} \left(K(y-x)\frac{\partial u(y)}{\partial \nu} - u(y)\frac{\partial K}{\partial \nu}(y-x) \right) d\sigma_y - \int_{\Omega} K(y-x)\Delta_y u(y)\, dy \tag{3.3.2}$$

For the Dirichlet problem, Δu is known in Ω and u is known on $\partial\Omega$. Thus, (3.3.2) gives an expression for the solution u, provided we know the normal derivative $\frac{\partial u(y)}{\partial \nu}$ along $\partial\Omega$. But this quantity is usually an unknown for Dirichlet problem. Thus, we wish to rewrite (3.3.2) such that the knowledge of the normal derivative is not necessary. To do so, we introduce a function $\psi_x(y)$, for a fixed $x \in \Omega$, as the solution of the boundary-value problem,

$$\begin{cases} \Delta \psi_x(y) &= 0 \text{ in } \Omega \\ \psi_x(y) &= K(y-x) \text{ on } \partial\Omega. \end{cases} \tag{3.3.3}$$

Now applying the second identity of Corollary C.0.8 for any $u \in C^2(\overline{\Omega})$ and $v(y) = \psi_x(y)$, we get

$$\int_{\partial\Omega} \left(u(y)\frac{\partial \psi_x(y)}{\partial \nu} - \psi_x(y)\frac{\partial u(y)}{\partial \nu} \right) d\sigma_y = \int_{\Omega} \left(u(y)\Delta_y \psi_x(y) - \psi_x(y)\Delta_y u(y) \right) dy.$$

Therefore, substituting the following identity

$$\int_{\partial\Omega} K(y-x)\frac{\partial u(y)}{\partial \nu}\, d\sigma_y = \int_{\Omega} \psi_x(y)\Delta_y u(y)\, dy + \int_{\partial\Omega} u(y)\frac{\partial \psi_x(y)}{\partial \nu}\, d\sigma_y$$

in (3.3.2), we get

$$u(x) = \int_{\Omega} \left(\psi_x(y) - K(y-x)\right)\Delta_y u(y)\, dy + \int_{\partial\Omega} u(y) \left(\frac{\partial \left(\psi_x(y) - K(y-x)\right)}{\partial \nu} \right) d\sigma_y.$$

The identity above motivates the definition of what is called the *Green's function*.

Definition 3.3.1. *For any given open subset $\Omega \subset \mathbb{R}^n$ and $x, y \in \Omega$ such that $x \neq y$, we define the Green's function as*

$$G(x,y) := \psi_x(y) - K(y-x).$$

Rewriting (3.3.2) in terms of Green's function,we get

$$u(x) = \int_\Omega G(x,y)\Delta_y u(y)\,dy + \int_{\partial\Omega} u(y)\frac{\partial G(x,y)}{\partial\nu}\,d\sigma_y.$$

Thus, in the arguments above we have proved the following theorem.

Theorem 3.3.2. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^1 boundary. Also, given $f \in C(\Omega)$ and $g \in C(\overline{\Omega})$. If $u \in C^2(\overline{\Omega})$ solves the Dirichlet problem* (3.3.1)*, then u has the representation*

$$u(x) = -\int_\Omega G(x,y)f(y)\,dy + \int_{\partial\Omega} g(y)\frac{\partial G(x,y)}{\partial\nu}\,d\sigma_y. \tag{3.3.4}$$

Observe that we have solved the Dirichlet problem (3.3.1) provided we know the Green's function. The construction of Green's function depends on the construction of ψ_x for every $x \in \Omega$. In other words, (3.3.1) is solved if we can solve (3.3.3). Ironically, computing ψ_x is usually possible when Ω has simple geometry. We shall identify two simple cases of Ω, half-space and balll, where we can explicitly compute G.

The Green's function is the analogue of the fundamental solution K for the boundary value problem. This is clear by observing that, for a fixed $x \in \Omega$, G satisfies (informally) the equation,

$$\begin{cases} -\Delta G(x,\cdot) &= \delta_x \text{ in } \Omega \\ G(x,\cdot) &= 0 \text{ on } \partial\Omega, \end{cases}$$

where δ_x is the Dirac measure at x.

Theorem 3.3.3. *For all $x, y \in \Omega$ such that $x \neq y$, we have $G(x,y) = G(y,x)$, i.e., G is symmetric in x and y.*

Proof. Let us fix $x, y \in \Omega$. For a fixed $m > 0$, set $\Omega_m = \Omega \setminus (B_m(x) \cup B_m(y))$

and applying Green's identity for $v(\cdot) := G(x,\cdot)$ and $w(\cdot) := G(y,\cdot)$, we get

$$\begin{aligned}
\int_{\partial\Omega_m}\left(v(z)\frac{\partial w(z)}{\partial\nu} - w(z)\frac{\partial v(z)}{\partial\nu}\right)d\sigma_z &= \int_{\Omega_m} v(z)\Delta_z w(z)\,dz \\
&\quad - \int_{\Omega_m} w(z)\Delta_z v(z)\,dz \\
\int_{\partial\Omega_m}\left(v(z)\frac{\partial w(z)}{\partial\nu} - w(z)\frac{\partial v(z)}{\partial\nu}\right)d\sigma_z &= 0 \\
\int_{S_m(x)}\left(v(z)\frac{\partial w(z)}{\partial\nu} - w(z)\frac{\partial v(z)}{\partial\nu}\right)d\sigma_z &= \int_{S_m(y)}\left(w(z)\frac{\partial v(z)}{\partial\nu} - v(z)\frac{\partial w(z)}{\partial\nu}\right)d\sigma_z \\
J_m(x) + K_m(x) &= J_m(y) + K_m(y).
\end{aligned}$$

$$\begin{aligned}
|J_m(x)| &\leq \int_{S_m(x)} |v(z)\nabla_z w(z)\cdot\nu|\,d\sigma_z \\
&\leq \|\nabla w\|_{\infty,\Omega}\int_{S_m(x)} |v(z)|\,d\sigma_z \\
&= \|\nabla w\|_{\infty,\Omega}\int_{S_m(x)} |\psi_x(z) - K(z-x)|\,d\sigma_z.
\end{aligned}$$

Thus, for $n = 2$,

$$|J_m(x)| \leq (2\pi m\|\psi_x\|_{\infty,\Omega} + m|\log m|)\,\|\nabla w\|_{\infty,\Omega}$$

and for $n \geq 3$, we have

$$|J_m(x)| \leq \left(\omega_n m^{n-1}\|\psi_x\|_{\infty,\Omega} + \frac{m}{(n-2)}\right)\|\nabla w\|_{\infty,\Omega}.$$

Hence, as $m \to 0$, $|J_m(x)| \to 0$. Now, consider the term $K_m(x)$,

$$\begin{aligned}
K_m(x) &= -\int_{S_m(x)} w(z)\frac{\partial v(z)}{\partial\nu}\,d\sigma_z \\
&= \int_{S_m(x)} w(z)\frac{\partial K}{\partial\nu}(z-x)\,d\sigma_z - \int_{S_m(x)} w(z)\frac{\partial\psi_x(z)}{\partial\nu}\,d\sigma_z.
\end{aligned}$$

The second term goes to zero by taking the sup-norm outside the integral. To tackle the first term, we note that $\nabla_z K(z-x) = \frac{-1}{\omega_n|z-x|^n}(z-x)$. Since we are in the m radius sphere $|z-x| = m$. Also the unit vector ν outside of $S_m(x)$, as a boundary of $\Omega\setminus B_m(x)$, is given by $-(z-x)/|z-x| = -(z-x)/m$. Therefore,

$$\nabla_z K(z-x)\cdot\nu = \frac{1}{\omega_n m^{n+1}}(z-x)\cdot(z-x) = \frac{1}{\omega_n m^{n-1}}.$$

$$\int_{S_m(x)} w(z)\nabla_z K(z-x)\cdot\nu\, d\sigma_z \quad = \quad \frac{1}{\omega_n m^{n-1}}\int_{S_m(x)} w(z)\, d\sigma_z$$

Since w is continuous in $\Omega\setminus\{y\}$, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|w(z) - w(x)| < \varepsilon$ whenever $|x-z| < \delta$. When $m \to 0$, we can choose m such that $m < \delta$ and for this m, we see that Now, consider

$$\left|\frac{1}{\omega_n m^{n-1}}\int_{S_m(x)} w(z)\, d\sigma_z - w(x)\right| \\ = \frac{1}{\omega_n m^{n-1}}\int_{S_m(x)} |w(z)-w(x)|\, d\sigma_z < \varepsilon.$$

Thus, as $m \to 0$, $K_m(x) \to w(x)$. Arguing similarly, for $J_m(y)$ and $K_m(y)$, we get $G(y,x) = G(x,y)$. □

3.3.2 Green's Function for half-space

In this section, we shall compute explicitly the Green's function for positive half-space. Thus, we shall have

$$\mathbb{R}^n_+ = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$$

and

$$\partial\mathbb{R}^n_+ = \{x = (x_1, x_2, \ldots, x_n\} \in \mathbb{R}^n \mid x_n = 0\}.$$

To compute the Green's function, we shall use the *method of reflection*. The reflection technique ensures that the points on the boundary (along which the reflection is done) remains unchanged to respect the imposed Dirichlet condition.

Definition 3.3.4. *For any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$, we define its* reflection *along its boundary $\mathbb{R}^{n-1}$ as $x^\star = (x_1, x_2, \ldots, -x_n)$.*

It is obvious from the above definition that, for any $y \in \partial\mathbb{R}^n_+$, $|y - x^\star| = |y - x|$. Given a fixed $x \in \mathbb{R}^n_+$, we need to find a harmonic function ψ_x in $\mathbb{R}^n_+$, as in (3.3.3). But $K(\cdot - x)$ is harmonic in $\mathbb{R}^n_+ \setminus \{x\}$. Thus, we use the method of reflection to shift the singularity of K from $\mathbb{R}^n_+$ to the negative half-space and define

$$\psi_x(y) = K(y - x^\star).$$

By definition, ψ_x is harmonic in $\mathbb{R}^n_+$ and on the boundary $\psi_x(y) = K(y - x)$. Therefore, we define the Green's function to be $G(x, y) = K(y - x^\star) - K(y - x)$, for all $x, y \in \mathbb{R}^n_+$ and $x \neq y$. It now only remains to compute the normal derivative of G. Recall that $\nabla_y K(y - x) = \frac{-1}{\omega_n |y-x|^n}(y - x)$. Thus,

$$\nabla_y G(x, y) = \frac{-1}{\omega_n}\left(\frac{y - x^\star}{|y - x^\star|^n} - \frac{y - x}{|y - x|^n}\right)$$

Therefore, when $y \in \partial\mathbb{R}^n_+$, we have

$$\nabla_y G(x, y) = \frac{-1}{\omega_n |y - x|^n}(x - x^\star).$$

Since the outward unit normal of $\partial\mathbb{R}^n_+$ is $\nu = (0, 0, \dots, 0, -1)$, we get

$$\nabla_y G(x, y) \cdot \nu = \frac{2x_n}{\omega_n |y - x|^n}.$$

Definition 3.3.5. *For all $x \in \mathbb{R}^n_+$ and $y \in \partial\mathbb{R}^n_+$, the map*

$$P(x, y) := \frac{2x_n}{\omega_n |y - x|^n}$$

is called the Poisson kernel *for $\mathbb{R}^n_+$.*

Now substituing for G in (3.3.4), we get the *Poisson formula* for u,

$$u(x) = \int_{\mathbb{R}^n_+} [K(y - x) - K(y - x^\star)] f(y)\, dy + \frac{2x_n}{\omega_n} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|y - x|^n}\, d\sigma_y. \tag{3.3.5}$$

It now remains to show that the u as defined above is, indeed, a solution of (3.3.1) for $\mathbb{R}^n_+$.

Exercise 6. Let $f \in C(\mathbb{R}^n_+)$ be given. Let $g \in C(\mathbb{R}^{n-1})$ be bounded. Then u as given in (3.3.5) is in $C^2(\mathbb{R}^n_+)$ and solves (3.3.1).

3.3.3 Green's Function for a disk

In this section, we shall compute explicitly the Green's function for a ball of radius $r > 0$ and centred at $a \in \mathbb{R}^n$, $B_r(a)$. As usual, we denote the surface of the disk as $S_r(a)$, the circle of radius r centred at a. We, once again, use the *method of reflection* but, this time reflected along the boundary of the disk.

Definition 3.3.6. *For any $x \in \mathbb{R}^n \setminus \{a\}$, we define its* reflection *along the circle $S_r(a)$ as $x^\star = \frac{r^2(x-a)}{|x-a|^2} + a$.*

The idea behind reflection is clear for the unit disk, i.e., when $a = 0$ and $r = 1$, as $x^\star = \frac{x}{|x|^2}$. The above definition is just the shift of origin to a and dilating the unit disk by r.

Now, for any $y \in S_r(a)$ and $x \neq a$, consider

$$\begin{aligned}
|y - x^\star|^2 &= |y-a|^2 - 2(y-a)\cdot(x^\star - a) + |x^\star - a|^2 \\
&= r^2 - 2r^2(y-a)\cdot\left(\frac{x-a}{|x-a|^2}\right) + \left|\frac{r^2(x-a)}{|x-a|^2}\right|^2 \\
&= \frac{r^2}{|x-a|^2}(|x-a|^2 - 2(y-a)\cdot(x-a) + r^2) \\
&= \frac{r^2}{|x-a|^2}(|x-a|^2 - 2(y-a)\cdot(x-a) + |y-a|^2) \\
&= \frac{r^2}{|x-a|^2}|y-x|^2
\end{aligned}$$

Therefore, $\frac{|x-a|}{r}|y-x^\star| = |y-x|$ for all $y \in S_r(a)$. For each fixed $x \in B_r(a)$, we need to find a harmonic function ψ_x in $B_r(a)$ solving (3.3.3). Since $K(\cdot - x)$ is harmonic in $B_r(a) \setminus \{x\}$, we use the method of reflection to shift the singularity of K at x to the complement of $B_r(a)$. Thus, we define

$$\psi_x(y) = K\left(\frac{|x-a|}{r}(y - x^\star)\right) \quad x \neq a.$$

For $n \geq 3$, $K\left(\frac{|x-a|}{r}(y-x^\star)\right) = \frac{|x-a|^{2-n}}{r^{2-n}}K(y-x^\star)$. Thus, for $n \geq 3$, ψ_x solves (3.3.3), for $x \neq a$. For $n = 2$,

$$K\left(\frac{|x-a|}{r}(y-x^\star)\right) = \frac{-1}{2\pi}\log\left(\frac{|x-a|}{r}\right) + K(y - x^\star).$$

Hence ψ_x solves (3.3.3) for $n = 2$. Note that we are yet to identify a harmonic function ψ_a corresponding to $x = a$. We do this by setting ψ_a to be the constant function

$$\psi_a(y) := \begin{cases} -\frac{1}{2\pi}\log r & (n = 2) \\ \frac{r^{2-n}}{\omega_n(n-2)} & (n \geq 3). \end{cases}$$

Thus, ψ_a is harmonic and solves (3.3.3) for $x = a$. Therefore, we define the Green's function to be

$$G(x,y) := K\left(\frac{|x-a|}{r}(y - x^\star)\right) - K(y-x) \quad \forall x, y \in B_r(a), x \neq a \text{ and } x \neq y$$

and

$$G(a,y) := \begin{cases} -\frac{1}{2\pi}\log\left(\frac{r}{|y-a|}\right) & (n = 2) \\ \frac{1}{\omega_n(n-2)}\left(r^{2-n} - |y-a|^{2-n}\right) & (n \geq 3). \end{cases}$$

We shall now compute the normal derivative of G. Recall that

$$\nabla_y K(y-x) = \frac{-1}{\omega_n|y-x|^n}(y-x)$$

and one can compute $\nabla_y K\left(\frac{|x-a|}{r}(y-x^\star)\right) = \frac{-|x-a|^{2-n}}{r^{2-n}\omega_n|y-x^\star|^n}(y-x^\star)$. Therefore,

$$\nabla_y G(x,y) = \frac{-1}{\omega_n}\left[\frac{|x-a|^{2-n}(y-x^\star)}{r^{2-n}|y-x^\star|^n} - \frac{y-x}{|y-x|^n}\right].$$

If $y \in S_r(a)$, we have

$$\begin{aligned} \nabla_y G(x,y) &= \frac{-1}{\omega_n|y-x|^n}\left[\frac{|x-a|^2}{r^2}(y-x^\star) - (y-x)\right] \\ &= \frac{-1}{\omega_n|y-x|^n}\left[\frac{|x-a|^2}{r^2} - 1\right](y-a) \end{aligned}$$

Since the outward unit normal at any point $y \in S_r(a)$ is $\frac{1}{r}(y-a)$, we have

$$\begin{aligned} \nabla_y G(x,y)\cdot\nu &= \frac{-1}{\omega_n|y-x|^n}\left[\frac{|x-a|^2}{r^2} - 1\right]\sum_{i=1}^n \frac{1}{r}(y_i - a_i)^2 \\ &= \frac{-r}{\omega_n|y-x|^n}\left[\frac{|x-a|^2}{r^2} - 1\right]. \end{aligned}$$

Definition 3.3.7. *For all $x \in B_r(a)$ and $y \in S_r(a)$, the map*

$$P(x, y) := \frac{r^2 - |x-a|^2}{r\omega_n |y-x|^n}$$

is called the Poisson kernel *for $B_r(a)$.*

Now substituing for G in (3.3.4), we get the *Poisson formula* for u,

$$u(x) = -\int_{B_r(a)} G(x,y) f(y)\,dy + \frac{r^2 - |x-a|^2}{r\omega_n} \int_{S_r(a)} \frac{g(y)}{|y-x|^n}\,d\sigma_y. \quad (3.3.6)$$

It now remains to show that the u as defined above is, indeed, a solution of (3.3.1) for $B_r(a)$.

Exercise 7. Let $f \in C(B_r(a))$ be given. Let $g \in C(S_r(a))$ be bounded. Then u as given in (3.3.6) is in $C^2(B_r(a))$ and solves (3.3.1).

3.3.4 Dirichlet principle

The Dirichlet principle (formulated, independently by Gauss, Lord Kelvin and Dirichlet) states that the solution of the Dirichlet problem minimizes the corresponding energy functional.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary $\partial\Omega$ and let $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ be given. For convenience, recall the Dirichlet problem ((3.3.1)),

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ \quad u &= g \quad \text{on } \partial\Omega. \end{cases}$$

Any solution u of (3.3.1) is in $V = \{v \in C^2(\overline{\Omega}) \mid v = g \text{ on } \partial\Omega\}$. The *energy functional* $J : V \to \mathbb{R}$ is defined as

$$J(v) := \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega f v\,dx$$

Theorem 3.3.8 (Dirichlet's principle)**.** *A $C^2(\overline{\Omega})$ function u solves* (3.3.1) *iff u minimises the functional J on V, i.e.,*

$$J(u) \le J(v) \quad \forall v \in V.$$

Proof. Let $u \in C^2(\overline{\Omega})$ be a solution of (3.3.1). For any $v \in V$, we multiply both sides of (3.3.1) by $u - v$ and integrating we get,

$$\begin{aligned}
\int_\Omega (-\Delta u - f)(u - v)\,dx &= 0 \\
\int_\Omega \nabla u \cdot \nabla(u - v)\,dx - \int_\Omega f(u - v)\,dx &= 0 \\
\int_\Omega \left(|\nabla u|^2 - fu\right)\,dx &= \int_\Omega (\nabla u \cdot \nabla v - fv)\,dx \\
&\leq \int_\Omega |\nabla u \cdot \nabla v| - \int_\Omega fv\,dx \\
&\leq \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right)\,dx - \int_\Omega fv\,dx \\
&\quad \text{(since } 2ab \leq a^2 + b^2 \text{)} \\
J(u) &\leq J(v).
\end{aligned}$$

Thus, u minimises J in V. Conversely, let u minimise J in V. Thus,

$$\begin{aligned}
J(u) &\leq J(v) \quad \forall v \in V \\
J(u) &\leq J(u + t\phi) \quad (\text{for any } \phi \in C^2(\Omega) \text{ such that } \phi = 0 \text{ on } \partial\Omega) \\
0 &\leq \frac{1}{t}\left(J(u + t\phi) - J(u)\right) \\
0 &\leq \frac{1}{t}\left(\frac{1}{2}\int_\Omega \left(t^2|\nabla\phi|^2 + 2t\nabla\phi \cdot \nabla u\right)\,dx - t\int_\Omega f\phi\,dx\right)
\end{aligned}$$

Taking limit $t \to 0$ both sides, we get

$$0 \leq \int_\Omega \nabla\phi \cdot \nabla u\,dx - \int_\Omega f\phi\,dx \quad \forall \phi \in C^2(\Omega) \text{ s.t. } \phi = 0 \text{ on } \partial\Omega.$$

Choosing $-\phi$ in place of ϕ we get the reverse inequality, and we have equality in the above. Thus,

$$\begin{aligned}
\int_\Omega \nabla u \cdot \nabla\phi\,dx &= \int_\Omega f\phi\,dx \quad \forall \phi \in C^2(\Omega) \text{ s.t. } \phi = 0 \text{ on } \partial\Omega \\
\int_\Omega (-\Delta u - f)\phi\,dx &= 0 \quad \forall \phi \in C^2(\Omega) \text{ s.t. } \phi = 0 \text{ on } \partial\Omega.
\end{aligned}$$

Thus u solves (3.3.1). □

3.4 Neumann Problem

The Neumann problem is stated as follows: Given $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$, find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} &= g \quad \text{on } \partial\Omega \end{cases} \tag{3.4.1}$$

where $\frac{\partial u}{\partial \nu} := \nabla u \cdot \nu$ and $\nu = (\nu_1, \ldots, \nu_n)$ is the outward pointing unit normal vector field of $\partial\Omega$. Thus, the boundary imposed is called the Neumann boundary condition. The solution of a Neumann problem is not necessarily unique. If u is any solution of (3.4.1), then $u + c$ for any constant c is also a solution of (3.4.1). More generally, for any v such that v is constant on the connected components of Ω, $u + v$ is a solution of (3.4.1). Moreover, if u is a solution of the Neumann problem (3.4.1) then u satisfies, for every connected component ω of Ω,

$$\begin{aligned} \int_\omega \Delta u &= \int_{\partial\omega} \nabla u \cdot \nu \quad \text{(Using GDT)} \\ \int_\omega f &= \int_{\partial\omega} g. \end{aligned}$$

The second equality is called the compatibility condition. Thus, if (3.4.1) is solvable then necessarily, the given function f, g must satisfy the compatibility condition.

3.5 Heat Equation

Let a homogeneous material occupy a region $\Omega \subset \mathbb{R}^n$ with C^1 boundary. Let k denote the thermal conductivity (dimensionless quantity) and c be the heat capacity of the material. Let $u(x,t)$ be a function plotting the temperature of the material at $x \in \Omega$ in time t. The thermal energy stored at $x \in \Omega$, at time t, is $cu(x,t)$. If $\mathbf{v}(x,t)$ denotes the velocity of (x,t), by Fourier law, the thermal energy changes following the gradients of temperature, i.e.,

$$cu(x,t)\mathbf{v}(x,t) = -k\nabla u.$$

The thermal energy is the quantity that is conserved (conservation law) and satisfies the continuity equation (1.2.1). Thus, we have

$$u_t - \frac{k}{c}\Delta u = 0.$$

If the material occupying the region Ω is non-homogeneous, anisotropic, the temperature gradient may generate heat in preferred directions, which themselves may depend on $x \in \Omega$. Thus, the conductivity of such a material at $x \in \Omega$, at time t, is given by a $n \times n$ matrix $K(x,t) = (k_{ij}(x,t))$. Thus, in this case, the heat equation becomes,

$$u_t - \operatorname{div}\left(\frac{1}{c} K \nabla u\right) = 0.$$

The heat equation is an example of a second order equation in divergence form. The heat equation gives the temperature distribution $u(x,t)$ of the material with conductivity k and capacity c. In general, we may choose $k/c = 1$, since, for any k and c, we may rescale our time scale $t \mapsto (k/c)t$.

3.5.1 Fundamental Solution

We shall now derive the fundamental solution of the heat equation

$$u_t(x,t) - \Delta u(x,t) = 0.$$

Note that if $u(x,t)$ is a solution of the heat equation, then $(u \circ T_\lambda)(x,t)$ is also a solution of the heat equation, where $T_\lambda(x,t) = (\lambda x, \lambda^2 t)$ is a linear transformation for any $\lambda \neq 0$. This scaling or dilation is called the parabolic scaling. Thus, we look for a solution $u(x,t) = v(t)w(r^2/t)$, where $r = |x|$. Substituting this separation of variable in the heat equation, we derive $v(t) = t^{-n/2}$ and $w(t) = e^{-r^2/4t}$. This motivates us to define the fundamental solution as

$$K(x,t) := \begin{cases} -\frac{1}{4\pi t}^{n/2} e^{-r^2/4t} & x \in \mathbb{R}^n, t > 0 \\ 0 & x \in \mathbb{R}^n, t < 0. \end{cases}$$

3.5.2 Duhamel's Principle

3.6 Wave Equation

3.6.1 D'Alembert's Formula

3.6.2 Method of Descent

3.6.3 Duhamel's Principle

Appendices

Appendix A

The Gamma Function

The gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ is defined as,

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt \quad \forall x \in (0, \infty).$$

Note that the gamma function is defined as an improper integral and its existence has to be justified. Observe that for a fixed $x > 0$,

$$|e^{-t} t^{x-1}| = e^{-t} t^{x-1} \le t^{x-1} \quad \forall t > 0,$$

since for $t > 0$, $|e^{-t}| \le 1$. Now, since $\int_0^1 t^{x-1} \, dt$ exists, we have by comparison test the existence of the integral $\int_0^1 e^{-t} t^{x-1} \, dt$. Now, for $t \to \infty$, $e^{-t} t^{x-1} \to 0$ and hence the is a constant $C > 0$ such that

$$t^{x-1} e^{-t} \le C/t^2 \quad \forall t \ge 1.$$

Since $\int_1^\infty 1/t^2 \, dt$ exists, we again have using comparison test the existence of the integral $\int_1^\infty e^{-t} t^{x-1} \, dt$. In fact, the gamma function can be defined for any complex number $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 0$.

It is worth noting that the gamma function Γ generalises the notion of factorial of positive integers. This would be the first property we shall prove.

Exercise 8. Show that $\Gamma(x + 1) = x\Gamma(x)$. In particular, for any positive integer n, $\Gamma(n + 1) = n!$. Also, show that $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$. Further, for any positive integer n,

$$\Gamma(n + 1/2) = (n - 1/2)(n - 3/2) \ldots (1/2)\sqrt{\pi}.$$

(Hint: Use integration by parts and change of variable).

Exercise 9. Show that Γ is continuous on $(0, \infty)$.

Exercise 10. Show that the logarithm of Γ is convex on $(0, \infty)$.

We shall now show that Γ is the only possible generalisation of the notion of factorial of positive integers.

Theorem A.0.1. *Let f be positive and continuous on $(0, \infty)$ and let $\log f$ be convex on $(0, \infty)$. Also, let f satisfy the recursive equation*

$$f(x+1) = xf(x) \quad \forall x > 0$$

and $f(1) = 1$, then $f(x) = \Gamma(x)$ for all $x > 0$.

Appendix B

Surface Area and Volume of Disk in $\mathbb{R}^n$

Theorem B.0.2 (Polar coordinates)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and integrable. Then*

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_0^\infty \left(\int_{S_r(a)} f(y)\,d\sigma_y \right) dr$$

for each $a \in \mathbb{R}^n$. In particular, for each $r > 0$,

$$\frac{d}{dr}\left(\int_{B_r(a)} f(x)\,dx \right) = \int_{S_r(a)} f(y)\,d\sigma_y.$$

Theorem B.0.3. *Prove that*

$$\int_{\mathbb{R}^n} e^{-\pi|x|^2}\,dx = 1.$$

Further, prove that the surface area ω_n of $S_1(0)$ in $\mathbb{R}^n$ is

$$\frac{2\pi^{n/2}}{\Gamma(n/2)}$$

and the volume of the ball $B_1(0)$ in $\mathbb{R}^n$ is ω_n/n. Consequently, for any $x \in \mathbb{R}^n$ and the $r > 0$, the surface area of $S_r(x)$ is $r^{n-1}\omega_n$ and the volume of $B_r(x)$ is $r^n\omega_n/n$.

Proof. We first observe that

$$e^{-\pi|x|^2} = e^{-\pi\left(\sum_{i=1}^n x_i^2\right)} = \prod_{i=1}^n e^{-\pi x_i^2}.$$

Therefore,

$$\begin{aligned} I_n := \int_{\mathbb{R}^n} e^{-\pi|x|^2}\,dx &= \int_{\mathbb{R}^n} \prod_{i=1}^n e^{-\pi x_i^2}\,dx \\ &= \prod_{i=1}^n \int_{\mathbb{R}} e^{-\pi t^2}\,dt \\ &= \left(\int_{\mathbb{R}} e^{-\pi t^2}\,dt\right)^n = (I_1)^n \end{aligned}$$

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-\pi|x|^2}\,dx &= \left(\int_{\mathbb{R}} e^{-\pi t^2}\,dt\right)^{2(n/2)} = \left((I_1)^2\right)^{n/2} = (I_2)^{n/2} \\ &= \left(\int_{\mathbb{R}^2} e^{-\pi|y|^2}\,dy\right)^{n/2} \\ &= \left(\int_0^{2\pi}\int_0^{\infty} e^{-\pi|y|^2}\,dy\right)^{n/2} \\ &= \left(\int_0^{2\pi}\int_0^{\infty} e^{-\pi r^2} r\,dr\,d\theta\right)^{n/2} \qquad \text{(since jacobian is } r\text{)} \\ &= \left(2\pi\int_0^{\infty} e^{-\pi r^2} r\,dr\right)^{n/2} \\ &= \left(\pi\int_0^{\infty} e^{-\pi s}\,ds\right)^{n/2} \qquad \text{(by setting } r^2 = s\text{)} \\ &= \left(\int_0^{\infty} e^{-q}\,dq\right)^{n/2} \qquad \text{(by setting } \pi s = q\text{)} \\ &= (\Gamma(1))^{n/2} = 1. \end{aligned}$$

Let ω_n denote the surface area of the unit sphere $S_1(0)$ in $\mathbb{R}^n$, i.e.,

$$\omega_n = \int_{S_1(0)} d\sigma,$$

where $d\sigma$ is the $n-1$-dimensional surface measure. Now, consider

$$\begin{aligned} 1 &= \int_{\mathbb{R}^n} e^{-\pi|x|^2}\,dx \\ &= \int_{S_1(0)} \int_0^\infty e^{-\pi r^2} r^{n-1}\,dr\,d\sigma \\ &= \omega_n \int_0^\infty e^{-\pi r^2} r^{n-1}\,dr \\ &= \frac{\omega_n}{2\pi^{n/2}} \int_0^\infty e^{-s} s^{(n/2)-1}\,ds \quad \text{(by setting } s = \pi r^2) \\ &= \frac{\omega_n \Gamma(n/2)}{2\pi^{n/2}}. \end{aligned}$$

Thus, $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$. We shall now compute the volume of the disk $B_1(0)$. Consider,

$$\int_{B_1(0)} dx = \omega_n \int_0^1 r^{n-1}\,dr = \frac{\omega_n}{n}.$$

For any $x \in \mathbb{R}^n$ and $r > 0$, we observe by the shifting of origin that the surface area of $S_r(x)$ is same as the surface area of $S_r(0)$. Let $S_r(0) = \{s \in \mathbb{R}^n \mid |s| = r\}$. Now

$$\int_{S_r(0)} d\sigma_s = \int_{S_1(0)} r^{n-1}\,d\sigma_t = r^{n-1}\omega_n,$$

where $t = s/r$. Thus, the surface area of $S_r(x)$ is $r^{n-1}\omega_n$. Similarly, volume of a disk $B_r(x)$ is $r^n\omega_n/n$. □

Appendix C

Divergence Theorem

Definition C.0.4. *For an open set $\Omega \subset \mathbb{R}^n$ we say that its boundary $\partial\Omega$ is C^k $(k \geq 1)$, if for every point $x \in \partial\Omega$, there is a $r > 0$ and a C^k diffeomorphism $\gamma : B_r(x) \to B_1(0)\mathbb{R}$ (i.e. γ^{-1} exists and both γ and γ^{-1} are k-times continuously differentiable) such that*

1. $\gamma(\partial\Omega \cap B_r(x)) \subset B_1(0) \cap \{x \in \mathbb{R}^n \mid x_n = 0\}$ *and*

2. $\gamma(\Omega \cap B_r(x)) \subset B_1(0) \cap \{x \in \mathbb{R}^n \mid x_n > 0\}$

We say $\partial\Omega$ is C^∞ if $\partial\Omega$ is C^k for all $k = 1, 2, \ldots$ and $\partial\Omega$ is analytic if γ is analytic.

Equivalently, a workable definition of C^k boundary would be the following: if for every point $x \in \partial\Omega$, there exists a neighbourhood B_x of x and a C^k function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\Omega \cap B_x = \{x \in B_x \mid x_n > \gamma(x_1, x_2, \ldots, x_{n-1})\}.$$

The divergence of a vector field is the measure of the magnitude (outgoing nature) of all source (of the vector field) and absorption in the region. The divergence theorem was discovered by C. F. Gauss in 1813[1] which relates the outward flow (flux) of a vector field through a closed surface to the behaviour of the vector field inside the surface (sum of all its "source" and "sink"). The divergence theorem is, in fact, the mathematical formulation of the conservation law.

[1] J. L. Lagrange seems to have discovered this, before Gauss, in 1762

Theorem C.0.5. *Let Ω be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary. If $v \in C^1(\overline{\Omega})$ then*

$$\int_\Omega \frac{\partial v}{\partial x_i}\, dx = \int_{\partial\Omega} v\nu_i \, d\sigma$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the outward pointing unit normal vector field and $d\sigma$ is the surface measure of $\partial\Omega$.

The domain Ω need not be bounded provided $|v|$ and $\left|\frac{\partial v}{\partial x_i}\right|$ decays as $|x| \to \infty$. The field of geometric measure theory attempts to identify the precise condition on $\partial\Omega$ and v for which divergence theorem or integration by parts hold.

Corollary C.0.6 (Integration by parts)**.** *Let Ω be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary. If $u, v \in C^1(\overline{\Omega})$ then*

$$\int_\Omega u\frac{\partial v}{\partial x_i}\, dx + \int_\Omega v\frac{\partial u}{\partial x_i}\, dx = \int_{\partial\Omega} uv\nu_i \, d\sigma.$$

Theorem C.0.7 (Gauss)**.** *Let Ω be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary. Given a vector field $V = (v_1, \ldots, v_n)$ on Ω such that $v_i \in C^1(\overline{\Omega})$ for all $1 \le i \le n$, then*

$$\int_\Omega \nabla \cdot V\, dx = \int_{\partial\Omega} V \cdot \nu \, d\sigma. \tag{C.0.1}$$

Corollary C.0.8 (Green's Identities)**.** *Let Ω be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary. Let $u, v \in C^2(\overline{\Omega})$, then*

(i)

$$\int_\Omega (v\Delta u + \nabla v \cdot \nabla u)\, dx = \int_{\partial\Omega} v\frac{\partial u}{\partial \nu}\, d\sigma,$$

where $\frac{\partial u}{\partial \nu} := \nabla u \cdot \nu$ and $\Delta := \nabla \cdot \nabla$.

(ii)

$$\int_\Omega (v\Delta u - u\Delta v)\, dx = \int_{\partial\Omega} \left(v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right) d\sigma.$$

Proof. Apply divergence theorem to $V = v\nabla u$ to get the first formula. To get second formula apply divergence theorem for both $V = v\nabla u$ and $V = u\nabla v$ and subtract one from the other. □

Appendix D

Mollifiers and Convolution

Exercise 11. Show that the Cauchy's exponential function, $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} \exp(-x^{-2}) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0, \end{cases}$$

is infinitely differentiable, i.e., is in $C^\infty(\mathbb{R})$.

Using the above Cauchy's exponential function, one can construct functions in $C_c^\infty(\mathbb{R}^n)$.

Exercise 12. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, show that $\rho : \mathbb{R}^n \to \mathbb{R}$ defined as

$$\rho(x) = \begin{cases} c\exp(\frac{-1}{1-|x|^2}) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

is in $C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(\rho) = B_1(0)$, ball with centre 0 and radius 1, where $c^{-1} = \int_{|x|\leq 1} \exp(\frac{-1}{1-|x|^2})\, dx$.

Thus, one can introduce a sequence of functions in $C_c^\infty(\mathbb{R}^n)$, called *mollifiers*. For $\varepsilon > 0$, we set

$$\rho_\varepsilon(x) = \begin{cases} c\varepsilon^{-n}\exp(\frac{-\varepsilon^2}{\varepsilon^2-|x|^2}) & \text{if } |x| < \varepsilon \\ 0 & \text{if } |x| \geq \varepsilon, \end{cases} \tag{D.0.1}$$

Equivalently, $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon)$.

Exercise 13. Show that $\rho_\varepsilon \geq 0$ and $\int_{\mathbb{R}^n} \rho_\varepsilon(x)\, dx = 1$ and is in $C_c^\infty(\mathbb{R}^n)$ with support in $B_\varepsilon(0)$.

Let $f, g \in L^1(\mathbb{R}^n)$. Their convolution $f * g$ is defined as, for $x \in \mathbb{R}^n$,

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy.$$

The integral on RHS is well-defined, since by Fubini's Theorem and the translation invariance of the Lebesgue measure, we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y)g(y)|\,dx\,dy = \int_{\mathbb{R}^n} |g(y)|\,dy \int_{\mathbb{R}^n} |f(x-y)|\,dx = \|g\|_1\|f\|_1 < +\infty.$$

Thus, for a fixed x, $f(x-y)g(y) \in L^1(\mathbb{R}^n)$.

Theorem D.0.9. *Let $\Omega \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ and let*

$$\Omega_\varepsilon := \{x \in \Omega \mid dist(x, \partial\Omega) > \varepsilon\}.$$

If $u : \Omega \to \mathbb{R}$ is locally integrable, i.e., for every compact subset $K \subset \Omega$, $\int_K |u| < +\infty$, then $u_\varepsilon := \rho_\varepsilon * u$ is in $C^\infty(\Omega_\varepsilon)$.

Proof. Fix $x \in \Omega_\varepsilon$. Consider

$$\begin{aligned}\frac{u_\varepsilon(x+he_i) - u_\varepsilon(x)}{h} &= \frac{1}{h}\int_\Omega [\rho_\varepsilon(x+he_i-y) - \rho_\varepsilon(x-y)]\,u(y)\,dy \\ &= \int_{B_\varepsilon(x)} \frac{1}{h}[\rho_\varepsilon(x+he_i-y) - \rho_\varepsilon(x-y)]u(y)\,dy.\end{aligned}$$

Now, taking $\lim_{h\to 0}$ both sides, we get

$$\begin{aligned}\frac{\partial u_\varepsilon(x)}{\partial x_i} &= \lim_{h\to 0}\int_{B_\varepsilon(x)} \frac{1}{h}[\rho_\varepsilon(x+he_i-y) - \rho_\varepsilon(x-y)]u(y)\,dy \\ &= \int_{B_\varepsilon(x)} \frac{\partial\rho_\varepsilon(x-y)}{\partial x_i}u(y)\,dy \\ &\qquad \text{(interchange of limits is due to the uniform convergence)} \\ &= \int_\Omega \frac{\partial\rho_\varepsilon(x-y)}{\partial x_i}u(y)\,dy = \frac{\partial\rho_\varepsilon}{\partial x_i} * u.\end{aligned}$$

Similarly, one can show that, for any tuple α, $D^\alpha u_\varepsilon(x) = (D^\alpha\rho_\varepsilon * u)(x)$. Thus, $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$. □

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